

# Poisson cohomology, Koszul duality, and Batalin-Vilkovisky algebras

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## Abstract

We study the noncommutative Poincaré duality between the Poisson homology and cohomology of a unimodular quadratic Poisson algebra and its Koszul dual, and that between the Hochschild homology and cohomology of their deformation quantizations. We show that Kontsevich's deformation quantization preserves the corresponding Poincaré duality, and as a corollary, the Batalin-Vilkovisky algebra structures that naturally arise in these cases are all isomorphic.

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# 1 Introduction

In this paper we study the noncommutative Poincaré duality between the Poisson homology and cohomology of a unimodular quadratic Poisson algebra and its Koszul dual, and that between the Hochschild homology and cohomology of their deformation quantizations, following the works of Shoikhet [32] and Dolgushev [9].

Let  $A = \mathbb{C}[x_1, \dots, x_n]$  be the complex polynomial algebra of  $n$  variables. Recall that a *Poisson structure* on  $A$  is a bivector  $\pi$  such that the Schouten bracket  $[\pi, \pi] = 0$ ; it is called *quadratic* if the coefficients of  $\pi$  are quadratic; that is,

$$\pi = \sum_{i,j} c_{ij} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}, \quad c_{ij} \in \mathbb{C}.$$

Quadratic Poisson algebras form an important class of Poisson algebras, and have been extensively studied in the past two decades. Several years ago, Shoikhet [32] observed that if  $\pi$  is quadratic, then the Koszul dual algebra  $A^!$  of  $A$ , namely  $A^! = \mathbf{\Lambda}(\xi_1, \dots, \xi_n)$  the graded symmetric algebra generated by  $n$  elements of degree  $-1$ , has a canonical Poisson structure  $\pi^!$  (let us call it the Koszul dual of  $\pi$ ), given by

$$\pi^! = \sum_{i,j} c_{ij} \xi_{j_1} \xi_{j_2} \frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{i_2}},$$

and proved that Kontsevich's deformation quantization preserves this type of Koszul duality. Shoikhet's result invokes the following natural question: *Does there exist any other property of a Poisson algebra that preserves or is preserved by Koszul duality?* This is the primary motivation of our study, and in this paper, we are mainly concerned with two important invariants of Poisson algebras, namely, the Poisson homology and cohomology.

The following theorem is clear from Shoikhet's article, once we explicitly write down the corresponding complexes.

**Theorem 1.1** (Theorem 3.4). *Let  $A = \mathbb{C}[x_1, \dots, x_n]$  be a quadratic Poisson algebra. Denote by  $A^!$  the Koszul dual Poisson algebra of  $A$ . Then there are isomorphisms*

$$\mathrm{HP}_\bullet(A) \cong \mathrm{HP}^{-\bullet}(A^!; A^!) \quad \text{and} \quad \mathrm{HP}^\bullet(A) \cong \mathrm{HP}^\bullet(A^!),$$

where  $A^i := \text{Hom}_{\mathbb{C}}(A^!, \mathbb{C})$  is the linear dual of  $A^!$ .

In the above theorem,  $\text{HP}_{\bullet}(-)$  is the Poisson homology,  $\text{HP}^{\bullet}(-)$  is the Poisson cohomology, and  $\text{HP}^{\bullet}(A^!; A^i)$  is the Poisson cohomology of  $A^!$  with values in its dual space.

Historically, the Poisson homology and cohomology were introduced by Koszul [20] and Lichnerowicz [24] respectively. They were further studied by Brylinski [3], Huebschmann [18], Weinstein and his school, and many others. In particular, in 1997 Weinstein [38] introduced the notion of *unimodular* Poisson manifolds, and two years later Xu [41] proved that in this case, there is a version of Poincaré duality

$$\text{HP}^{\bullet}(M) \cong \text{HP}_{n-\bullet}(M), \quad (1)$$

where  $M$  is an  $n$ -dimensional oriented Poisson manifold. A purely algebraic version of Weinstein's notion was formulated later by Dolgushev in [9]; see also [22, 27], etc.

There is another version of Poincaré duality for the Poisson homology and cohomology of *finite dimensional* Poisson algebras such as  $A^!$  above. This was studied by Zhu, Van Oystaeyen and Zhang in [42]. More precisely, a finite dimensional (possibly graded) Poisson algebra, say  $A^!$ , is called *Frobenius* if it is equipped with a cyclically invariant non-degenerate pairing. In the rest of the paper, we shall use the word *cyclic* instead of *Frobenius*, just to be consistent with other references. They proved that if a cyclic Poisson algebra is *unimodular* in some sense (to be recalled below), then there exists Poincaré duality in this case:

$$\text{HP}^{\bullet}(A^!) \cong \text{HP}^{\bullet-n}(A^!; A^i), \quad (2)$$

where  $n$  is the degree of the pairing.

Combining the two versions of Poincaré duality (1) and (2) as well as Theorem 1.1, we have the following:

**Theorem 1.2** (Theorems 4.16+4.17). *Let  $A = \mathbb{C}[x_1, \dots, x_n]$  be a quadratic Poisson algebra. Then  $(A, \pi)$  is unimodular if and only if its Koszul dual  $(A^!, \pi^!)$  is unimodular cyclic. In this case, we have the following commutative diagram:*

$$\begin{array}{ccc} \text{HP}^{\bullet}(A) & \xrightarrow{\cong} & \text{HP}_{n-\bullet}(A) \\ \downarrow \cong & & \downarrow \cong \\ \text{HP}^{\bullet}(A^!) & \xrightarrow{\cong} & \text{HP}^{\bullet-n}(A^!; A^i). \end{array}$$

The main technique to prove the above theorem is the so-called “differential calculus”, a notion introduced by Tamarkin and Tsygan in [33]. Later, Lambre [21] used the terminology “differential calculus with duality” to study the noncommutative Poincaré duality in these cases.

In the above-mentioned two references [41, 42], the authors also proved that the Poisson cohomology of a unimodular Poisson algebra (in both cases) has a Batalin-Vilkovisky algebra structure. Batalin-Vilkovisky structure is a very important algebraic structure that has appeared in, for example, mathematical physics, Calabi-Yau geometry and string topology. For unimodular quadratic Poisson algebras, we have the following:

**Theorem 1.3** (Theorem 5.5). *Suppose  $A = \mathbb{C}[x_1, \dots, x_n]$  is unimodular quadratic Poisson algebra. Denote by  $A^!$  the Koszul dual Poisson algebra of  $A$ . Then*

$$\mathrm{HP}^\bullet(A) \cong \mathrm{HP}^\bullet(A^!)$$

*is an isomorphism of Batalin-Vilkovisky algebras.*

The above theorems have some analogy to the case of Calabi-Yau algebras. Calabi-Yau algebras were introduced by Ginzburg [16] in 2006. Ever since then, they have been widely studied by many mathematicians from various directions. In the past decade an important class of Calabi-Yau that are well studied are those which are Koszul. In this case, the Koszul dual of a Koszul Calabi-Yau algebra is a cyclic algebra. In [16, §5.4] Ginzburg stated a conjecture, which he attributed to R. Rouquier, saying that for a Koszul Calabi-Yau algebra, say  $A$ , its Hochschild cohomology is isomorphic to the Hochschild cohomology of its Koszul dual  $A^!$

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(A^!) \quad (3)$$

as Batalin-Vilkovisky algebras. This conjecture is recently proved by two authors of the current paper together with G. Zhou in [6]. In fact, Theorem 1.3 may be viewed as a generalization of Rouquier's conjecture in Poisson geometry, which has been folklore for several years.

More than being just an analogy, in [9, Theorem 3], Dolgushev proved that for the coordinate ring  $A$  of an affine Calabi-Yau Poisson variety, its deformation quantization in the sense of Kontsevich [19], say  $A_\hbar$ , is Calabi-Yau if and only if  $A$  is unimodular. Similarly in the paper [13], Felder and Shoikhet proved that for a cyclic Poisson algebra, its deformation quantization is again cyclic if and only if it is unimodular (see loc. cit. Corollary in page 77 and also Willwacher-Calaque [40, Theorem 37]). Dolgushev then asked two questions in [9, §7]. The first question is whether there is any relationship between these two types of unimodularity appeared in Kontsevich's deformation quantization. If we generalize Kontsevich's deformation quantization to the graded Poisson algebra case (as has been done by Cattaneo and Felder in [5]), then the following theorem partially answers his question, although both cases that Dolgushev and Felder-Shoikhet considered are more general:

**Theorem 1.4** (Theorem 7.6). *Let  $A = \mathbb{C}[x_1, \dots, x_n]$  be a quadratic Poisson algebra. Denote by  $A^!$  the Koszul dual Poisson algebra of  $A$ , and by  $A_\hbar$  and  $A_\hbar^!$  the Kontsevich deformation quantization of  $A$  and  $A^!$  respectively. If  $A$  is unimodular (recall that by Theorem 1.2  $A^!$  is unimodular cyclic), then  $A_\hbar$  is Calabi-Yau and  $A_\hbar^!$  is cyclic, and the following diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A[[\hbar]]) & \xrightarrow{\cong} & \mathrm{HP}^\bullet(A^![[\hbar]]) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^\bullet(A_\hbar) & \xrightarrow{\cong} & \mathrm{HH}^\bullet(A_\hbar^!). \end{array} \quad (4)$$

*is commutative as Batalin-Vilkovisky algebra isomorphisms, where  $A[[\hbar]]$  and  $A^![[\hbar]]$  are equipped with the Poisson bivectors  $\hbar\pi$  and  $\hbar\pi^!$  respectively.*

In other words, the first half of the theorem says that, under Koszul duality, the unimodularity that guarantees the deformation quantization of a Poisson Calabi-Yau algebra to be again Calabi-Yau becomes the unimodularity that guarantees the deformation quantization of a Poisson cyclic algebra to be again cyclic. More explicitly, in Kontsevich's deformation quantization of Poisson algebras, the two versions of unimodularity that appeared in [9] and [13] are related by Koszul duality. Note that in the above theorem,  $A_\hbar$  and  $A_\hbar^!$  are Koszul dual to each other by Shoikhet [32, Theorem 0.3].

The second question that Dolgushev asked in [9, §7] is whether there exists a relationship between the Poincaré duality of the Poisson (co)homology of  $A$  and the Poincaré duality of the Hochschild (co)homology of  $A_\hbar$ . The following theorem, on which the proof of the second half of Theorem 1.4 is based, answers this question:

**Theorem 1.5** (Theorems 7.2+7.5). (1) *Suppose  $A = \mathbb{C}[x_1, \dots, x_n]$  is a unimodular Poisson algebra. Let  $A_\hbar$  be its deformation quantization. Then the following diagram*

$$\begin{array}{ccc} \mathrm{HH}^\bullet(A_\hbar) & \xrightarrow{\cong} & \mathrm{HH}_{n-\bullet}(A_\hbar) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HP}^\bullet(A[[\hbar]]) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A[[\hbar]]) \end{array}$$

*is commutative.*

(2) *Similarly, suppose  $A^! = \Lambda(\xi_1, \dots, \xi_n)$  is a unimodular cyclic Poisson algebra. Let  $A_\hbar^!$  be its deformation quantization. Then the following diagram*

$$\begin{array}{ccc} \mathrm{HH}^\bullet(A_\hbar^!) & \xrightarrow{\cong} & \mathrm{HH}_{\bullet-n}(A_\hbar^!; A_\hbar^!) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HP}^\bullet(A^![[\hbar]]) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^![[\hbar]]; A^![[\hbar]]) \end{array}$$

*is commutative.*

The theorem says that, the two versions of Poincaré duality, one between the Poisson cohomology and homology, and the other between the Hochschild cohomology and homology, are preserved under Kontsevich's deformation quantization. Thus as a corollary, one obtains that *if  $A = \mathbb{C}[x_1, \dots, x_n]$  is a unimodular quadratic Poisson algebra, then all the homology and cohomology groups (Poisson and Hochschild) appeared in Theorems 1.4 and 1.5 are isomorphic.*

The rest of the paper is devoted to the proof of above theorems. It is organized as follows: in §2 we collect several facts on Koszul algebras, and their application to quadratic Poisson polynomials; in §3 we first recall the definition of Poisson homology and cohomology, and then prove Theorem 1.1; in §4 we study unimodular quadratic Poisson algebras and their Koszul dual, and prove Theorem 1.2; in §5 we prove Theorem 1.3 by means of the so-called “differential calculus with duality”; in §6 we discuss Calabi-Yau algebra, their Koszul duality and the Batalin-Vilkovisky algebras associated to them; at last, in §7 we discuss the deformation quantization of Poisson algebras and prove Theorems 1.4 and 1.5.

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**Convention.** Throughout the paper,  $k$  is an algebraically closed field of characteristic zero, which we may assume to be  $\mathbb{C}$ . All tensors and morphisms are graded over  $k$  unless otherwise specified. For a chain complex, its homology is denoted by  $H_\bullet(-)$ , and its cohomology is  $H^\bullet(-) := H_{-\bullet}(-)$ .

## 2 Preliminaries on Koszul algebras

In this section, we collect some necessary facts about Koszul algebras. The best reference is the monograph of Loday-Vallette [26]. The main result of this section is Lemma 2.3, which is due to Shoikhet [32], saying that for a quadratic Poisson polynomial algebra, its Koszul dual algebra admits a canonical Poisson structure.

### 2.1 Quadratic and Koszul algebras

Let  $V$  be a finite-dimensional (possibly graded) vector space over  $k$ . Denote by  $TV$  the free algebra generated by  $V$  over  $k$ ; that is,  $TV$  is the tensor algebra generated by  $V$ . Suppose  $R$  is a subspce of  $V \otimes V$ , and let  $(R)$  be the two-sided ideal generated by  $R$  in  $TV$ , then the quotient algebra  $A := TV/(R)$  is called a *quadratic algebra*. There are two concepts associated to a quadratic algebra, namely, its *Koszul dual coalgebra* and *Koszul dual algebra*, which are given as follows:

(1) Consider the subspace

$$U = \bigoplus_{n=0}^{\infty} U_n := \bigoplus_{n=0}^{\infty} \bigcap_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}$$

of  $TV$ , then  $U$  is not an algebra, but a coalgebra, whose coproduct is induced from the deconcatenation of the tensor products. The *Koszul dual coalgebra* of  $A$ , denoted by  $A^i$ , is

$$A^i = \bigoplus_{n=0}^{\infty} \Sigma^{\otimes n}(U_n),$$

where  $\Sigma$  is the degree shifting-up (suspension) functor.  $A^i$  naturally has a graded coalgebra structure induced from that of  $U$ ; for example, if all elements of  $V$  have degree zero, then

$$(A^i)_0 = k, \quad (A^i)_1 = V, \quad (A^i)_2 = R, \quad \dots\dots$$

(2) The *Koszul dual algebra* of  $A$ , denoted by  $A^!$ , is just the linear dual space of  $A^i$ , which is then a graded algebra. More precisely, Let  $V^* = \text{Hom}(V, k)$  be the linear dual space of  $V$ , and let  $R^\perp$  denote the space of annihilators of  $R$  in  $V^* \otimes V^*$ . Shift the grading of  $V^*$  down by one, denoted by  $\Sigma^{-1}V^*$ , then

$$A^! = T(\Sigma^{-1}V^*)/(\Sigma^{-1} \otimes \Sigma^{-1} \circ R^\perp).$$

Choose a set of basis  $\{e_i\}$  for  $V$ , and let  $\{e_i^*\}$  be their duals in  $V^*$ . There is a natural chain complex associated to  $A$ , called the *Koszul complex*:

$$\cdots \xrightarrow{\delta} A \otimes A_{i+1}^i \xrightarrow{\delta} A \otimes A_i^i \xrightarrow{\delta} \cdots \longrightarrow A \otimes A_0^i \xrightarrow{\delta} k, \quad (5)$$

where for any  $r \otimes f \in A \otimes A_i^i$ ,  $\delta(r \otimes f) = \sum_i e_i r \otimes \Sigma^{-1} e_i^* f$ .

**Definition 2.1** (Koszul algebra). A quadratic algebra  $A = TV/(R)$  is called *Koszul* if the Koszul chain complex (5) is acyclic.

**Example 2.2** (Polynomials). Let  $A = k[x_1, x_2, \dots, x_n]$  be the space of polynomials (the symmetric tensor algebra) with  $n$  generators, with each  $x_i$  having degree zero. It is well-known (cf. [26]) that  $A$  is a Koszul algebra, and its Koszul dual algebra  $A^!$  is the graded symmetric algebra  $\Lambda(\xi_1, \xi_2, \dots, \xi_n)$ , with grading  $|\xi_i| = -1$ .

## 2.2 Koszul duality for quadratic Poisson polynomial algebras

Recall that a *Poisson bracket* on a (possibly graded) commutative algebra  $A$  is a Lie bracket

$$[-, -] : A \otimes A \longrightarrow A,$$

which is a derivation for each argument.

Take  $A = k[x_1, x_2, \dots, x_n]$  with each  $|x_i| = 0$ . The Poisson bracket is given in the form of the Poisson bivector

$$\pi = \sum_{i,j} \phi_{ij}(x_1, \dots, x_n) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where  $\phi_{ij}(x_1, \dots, x_n)$  are polynomials. The Poisson algebra  $(A, \pi)$  is called *quadratic* if  $\phi_{ij}$  are all quadratics, that is,

$$\pi = \sum_{i,j} c_{ij} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}, \quad c_{ij} \in k. \quad (6)$$

In this case, the Poisson bivector  $\pi$  gives a bivector  $\pi^!$  on  $A^!$ , which we would call the Koszul dual of  $\pi$ , and is given by

$$\pi^! := \sum_{i,j} c_{ij} \xi_{j_1} \xi_{j_2} \frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{i_2}} \quad (7)$$

**Lemma 2.3** (Shoikhet [32]). *Let  $A = k[x_1, \dots, x_n]$  with a bivector  $\pi$  in the form (6). Then  $(A, \pi)$  is quadratic Poisson if and only if  $(A^!, \pi^!)$  is quadratic Poisson.*

*Proof.* This is a key observation in Shoikhet [32]. Since it is so important, we give a proof here for completeness.

A bivector  $\pi$  defines a Poisson structure if and only if  $[\pi, \pi] = 0$ . Plugging (6) in  $[\pi, \pi]$  we have

$$[\pi, \pi] = \sum_{i_1, i_2, j_1, j_2} \sum_{k_1, k_2, \ell_1, \ell_2} \left( c_{ij} c_{kl} x_{i_1} x_{i_2} (\delta_{k_1 j_2} x_{k_2} + \delta_{k_2 j_2} x_{k_1}) \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{\ell_1}} \wedge \frac{\partial}{\partial x_{\ell_2}} \right. \quad (8a)$$

$$\left. - c_{ij} c_{kl} x_{i_1} x_{i_2} (\delta_{k_1 j_1} x_{k_2} + \delta_{k_2 j_1} x_{k_1}) \frac{\partial}{\partial x_{j_2}} \wedge \frac{\partial}{\partial x_{\ell_1}} \wedge \frac{\partial}{\partial x_{\ell_2}} \right) \quad (8b)$$

$$+c_{k\ell}c_{ij}x_{k_1}x_{k_2}(\delta_{i_1\ell_2}x_{i_2} + \delta_{i_2\ell_2}x_{i_1})\frac{\partial}{\partial x_{\ell_1}} \wedge \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}} \quad (8c)$$

$$-c_{k\ell}c_{ij}x_{k_1}x_{k_2}(\delta_{i_1\ell_1}x_{i_2} + \delta_{i_2\ell_1}x_{i_1})\frac{\partial}{\partial x_{\ell_2}} \wedge \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}), \quad (8d)$$

where  $\delta_{\bullet\bullet}$  is the Kronecker symbol. Similarly, plugging the formula (7) into  $[\pi^!, \pi^!]$ , we obtain

$$[\pi^!, \pi^!] = \sum_{i_1, i_2, j_1, j_2} \sum_{k_1, k_2, \ell_1, \ell_2} \left( c_{ij}c_{k\ell}\xi_{j_1}\xi_{j_2}(\delta_{\ell_1 i_2}\xi_{\ell_2} - \delta_{\ell_2 i_2}\xi_{\ell_1})\frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{k_1}} \wedge \frac{\partial}{\partial \xi_{k_2}} \right. \quad (9a)$$

$$\left. + c_{ij}c_{k\ell}\xi_{j_1}\xi_{j_2}(\delta_{\ell_1 i_1}\xi_{\ell_2} - \delta_{\ell_2 i_1}\xi_{\ell_1})\frac{\partial}{\partial \xi_{i_2}} \wedge \frac{\partial}{\partial \xi_{k_1}} \wedge \frac{\partial}{\partial \xi_{k_2}} \right. \quad (9b)$$

$$\left. + c_{k\ell}c_{ij}\xi_{\ell_1}\xi_{\ell_2}(\delta_{j_1 k_2}\xi_{j_2} - \delta_{j_2 k_2}\xi_{j_1})\frac{\partial}{\partial \xi_{k_1}} \wedge \frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{i_2}} \right. \quad (9c)$$

$$\left. + c_{k\ell}c_{ij}\xi_{\ell_1}\xi_{\ell_2}(\delta_{j_1 k_1}\xi_{j_2} - \delta_{j_2 k_1}\xi_{j_1})\frac{\partial}{\partial \xi_{k_2}} \wedge \frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{i_2}} \right). \quad (9d)$$

Under the correspondence

$$x_{i_{1,2}} \mapsto \frac{\partial}{\partial \xi_{i_{1,2}}}, \quad x_{k_{1,2}} \mapsto \frac{\partial}{\partial \xi_{k_{1,2}}}, \quad \frac{\partial}{\partial x_{j_{1,2}}} \mapsto \xi_{j_{1,2}}, \quad \frac{\partial}{\partial x_{\ell_{1,2}}} \mapsto \xi_{\ell_{1,2}}, \quad (10)$$

we see that the summands

$$(8a) + (8b) = (9c) + (9d) \quad \text{and} \quad (8c) + (8d) = (9a) + (9b),$$

and therefore,  $[\pi, \pi] = 0$  if and only if  $[\pi^!, \pi^!] = 0$ . This completes the proof.  $\square$

### 2.3 Koszul algebra over a discrete evaluation ring

So far, we have assumed that  $V$  is a  $k$ -linear space. In §7, we will study the deformed algebras, which are algebras over  $k[[\hbar]]$ . In [32], Shoikhet proved that results in above subsections remain to hold for algebras over a discrete evaluation ring, such as  $k[[\hbar]]$ . For example,  $k[x_1, \dots, x_n][[\hbar]]$  is Koszul dual to  $\mathbf{A}(\xi_1, \dots, \xi_n)[[\hbar]]$  as graded algebras over  $k[[\hbar]]$  (see [32, Theorem 0.3]).

## 3 Poisson homology and cohomology

The notions of Poisson homology and cohomology were introduced by Koszul [20] and Lichnerowicz [24] respectively. For more details of these concepts, we recommend the new classic [23] of Laurent-Gengoux, Pichereau and Vanhaecke.

### 3.1 Definitions

**Definition 3.1** (Poisson homology; Koszul [20]). Suppose  $(A, \pi)$  is a Poisson algebra. Denote by  $\Omega^p(A)$  the set of  $p$ -th Kähler differential forms of  $A$ . Then the *Poisson chain complex* of  $A$ , denoted by  $\text{CP}_\bullet(A, \pi)$ , is

$$\dots \longrightarrow \Omega^{p+1}(A) \xrightarrow{\partial} \Omega^p(A) \xrightarrow{\partial} \Omega^{p-1}(A) \xrightarrow{\partial} \dots \longrightarrow \Omega^0(A) = A, \quad (11)$$



where  $\partial$  is given by

$$\begin{aligned}\partial(f_0 df_1 \wedge \cdots \wedge df_p) &= \sum_{i=1}^p (-1)^{i-1} \{f_0, f_i\} df_1 \wedge \cdots \widehat{df_i} \cdots \wedge df_p \\ &+ \sum_{1 \leq i < j \leq p} (-1)^{j-i} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \cdots \widehat{df_i} \cdots \widehat{df_j} \cdots \wedge df_p.\end{aligned}$$

The associated homology is called the *Poisson homology* of  $A$ , and is denoted by  $\mathrm{HP}_\bullet(A, \pi)$ .

**Definition 3.2** (Poisson cohomology; Lichnerowicz [24]). Suppose  $(A, \pi)$  is a Poisson algebra and  $M$  is a left Poisson  $A$ -module. Let  $\mathfrak{X}_A^{-p}(M)$  be the space of skew-symmetric multilinear maps  $A^{\otimes p} \rightarrow M$  that are derivations in each argument. The *Poisson cochain complex* of  $A$  with values in  $M$ , denoted by  $\mathrm{CP}^\bullet(A, \pi; M)$ , is the cochain complex

$$M = \mathfrak{X}_A^0(M) \xrightarrow{\delta} \cdots \longrightarrow \mathfrak{X}_A^{-p+1}(M) \xrightarrow{\delta} \mathfrak{X}_A^{-p}(M) \xrightarrow{\delta} \mathfrak{X}_A^{-p-1}(M) \xrightarrow{\delta} \cdots$$

where  $\delta$  is given by

$$\begin{aligned}\delta(P)(f_0, f_1, \dots, f_p) &:= \sum_{0 \leq i \leq p} (-1)^i \{f_i, P(f_0, \dots, \widehat{f_i}, \dots, f_p)\} \\ &+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} P(\{f_i, f_j\}, f_1, \dots, \widehat{f_i}, \dots, \widehat{f_j}, \dots, f_p).\end{aligned}$$

The associated cohomology is called the *Poisson cohomology* of  $A$  with values in  $M$ , and is denoted by  $\mathrm{HP}^\bullet(A, \pi; M)$ . In particular, if  $M = A$ , then the cohomology is just called the *Poisson cohomology* of  $A$ , and is simply denoted by  $\mathrm{HP}^\bullet(A, \pi)$ .

Note that in the above definition, the Poisson cochain complex, viewed as a chain complex, is negatively graded, and the coboundary  $\delta$  has degree  $-1$ . However, by our convention, the Poisson cohomology are positively graded. In the following, if the Poisson structure is clear from the context, we often omit  $\pi$  in the notations of Poisson homology and cohomology.

From the universal property of Kähler differentials (*cf.* Loday [25, Proposition 1.3.9]), there is an identity of left  $A$ -modules

$$\mathfrak{X}_A^{-p}(M) = \mathrm{Hom}_A(\Omega^p(A), M). \quad (12)$$

**Remark 3.3** (The graded case). The Poisson homology and cohomology can be defined for graded Poisson algebras as well. In this case,

$$\Omega^p(A) = \bigoplus_{n \in \mathbb{Z}} \left\{ f_0 df_1 \wedge \cdots \wedge df_n \mid f_i \in A, |f_0| + |f_1| + \cdots + |f_n| + n = p \right\}$$

and  $\mathfrak{X}_A^{-p}(M)$  is again given by  $\mathrm{Hom}_A(\Omega^p(A), M)$ . The boundary maps are completely analogous to those of Poisson chain and cochain complexes (with Koszul's sign convention counted).

### 3.2 Isomorphism of the Poisson cohomology for Koszul pairs

We are now ready to prove Theorem 1.1.

**Theorem 3.4** (Theorem 1.1). *Let  $A = k[x_1, \dots, x_n]$ . Suppose  $(A, \pi)$  is a quadratic Poisson algebra, and denote by  $(A^!, \pi^!)$  its Koszul dual. Then there are isomorphisms*

$$\mathrm{HP}_\bullet(A, \pi) \cong \mathrm{HP}^{-\bullet}(A^!, \pi^!; A^!), \quad \mathrm{HP}^\bullet(A, \pi) \cong \mathrm{HP}^\bullet(A^!, \pi^!). \quad (13)$$

*Proof.* (1) We first show the first isomorphism in (13). Since  $A = k[x_1, \dots, x_n]$ , we have an explicit expression for  $\Omega^\bullet(A)$ , which is

$$\Omega^\bullet(A) = \mathbf{\Lambda}(x_1, \dots, x_n, dx_1, \dots, dx_n), \quad (14)$$

where  $\mathbf{\Lambda}$  means the graded symmetric tensor product, and  $|x_i| = 0$  and  $|dx_i| = 1$ , for  $i = 1, \dots, n$ . Similarly,

$$\Omega^\bullet(A^!) = \mathbf{\Lambda}(\xi_1, \dots, \xi_n, d\xi_1, \dots, d\xi_n),$$

where  $|\xi_i| = -1$  and  $|d\xi_i| = 0$  for  $i = 1, \dots, n$ , and therefore

$$\begin{aligned} \mathfrak{X}_{A^!}^\bullet(A^!) &= \mathrm{Hom}_{A^!}(\Omega^\bullet(A^!), A^!) \\ &= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1, \dots, \xi_n)}(\mathbf{\Lambda}(\xi_1, \dots, \xi_n, d\xi_1, \dots, d\xi_n), \mathrm{Hom}(\mathbf{\Lambda}(\xi_1, \dots, \xi_n), k)) \\ &= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1, \dots, \xi_n)}(\mathbf{\Lambda}(\xi_1, \dots, \xi_n) \otimes \mathbf{\Lambda}(d\xi_1, \dots, d\xi_n), \mathrm{Hom}(\mathbf{\Lambda}(\xi_1, \dots, \xi_n), k)) \\ &= \mathrm{Hom}(\mathbf{\Lambda}(d\xi_1, \dots, d\xi_n), \mathrm{Hom}(\mathbf{\Lambda}(\xi_1, \dots, \xi_n), k)) \\ &= \mathrm{Hom}(\mathbf{\Lambda}(d\xi_1, \dots, d\xi_n) \otimes \mathbf{\Lambda}(\xi_1, \dots, \xi_n), k) \\ &= \mathrm{Hom}(\mathbf{\Lambda}(d\xi_1, \dots, d\xi_n, \xi_1, \dots, \xi_n), k) \\ &= \mathbf{\Lambda}\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}, \xi_1^*, \dots, \xi_n^*\right). \end{aligned} \quad (15)$$

Thus from (14) and (15) there is a canonical grading preserving isomorphism of vector spaces:

$$\begin{aligned} \Phi : \Omega^\bullet(A) &\longrightarrow \mathfrak{X}^\bullet(A^!) \\ x_i &\longmapsto \frac{\partial}{\partial \xi_i} \\ dx_i &\longmapsto \xi_i^*, \quad i = 1, \dots, n. \end{aligned}$$

It is a direct check that  $\Phi$  is a chain map, and thus we obtain a quasiisomorphism of Poisson complexes

$$\Phi : \mathrm{CP}_\bullet(A, \pi) \cong \mathrm{CP}^{-\bullet}(A^!, \pi^!; A^!), \quad (16)$$

which then induces an isomorphism on homology.

(2) We now show the second isomorphism in (13). Similar to the above argument, we have

$$\begin{aligned} \mathrm{CP}^\bullet(A) &= \mathrm{Hom}_A(\Omega^\bullet(A), A) \\ &= \mathrm{Hom}_{\mathbf{\Lambda}(x_1, \dots, x_n)}(\mathbf{\Lambda}(x_1, \dots, x_n, dx_1, \dots, dx_n), \mathbf{\Lambda}(x_1, \dots, x_n)) \\ &= \mathrm{Hom}_{\mathbf{\Lambda}(x_1, \dots, x_n)}(\mathbf{\Lambda}(x_1, \dots, x_n) \otimes \mathbf{\Lambda}(dx_1, \dots, dx_n), \mathbf{\Lambda}(x_1, \dots, x_n)) \\ &= \mathrm{Hom}(\mathbf{\Lambda}(dx_1, \dots, dx_n), \mathbf{\Lambda}(x_1, \dots, x_n)) \\ &= \mathbf{\Lambda}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \otimes \mathbf{\Lambda}(x_1, \dots, x_n) \end{aligned} \quad (17)$$

and similarly,

$$\mathrm{CP}^\bullet(A^!) = \mathrm{Hom}_{A^!}(\Omega^\bullet(A^!), A^!)$$

$$\begin{aligned}
&= \text{Hom}_{\Lambda(\xi_1, \dots, \xi_n)}(\Lambda(\xi_1, \dots, \xi_n, d\xi_1, \dots, d\xi_n), \Lambda(\xi_1, \dots, \xi_n)) \\
&= \text{Hom}_{\Lambda(\xi_1, \dots, \xi_n)}(\Lambda(\xi_1, \dots, \xi_n) \otimes \Lambda(d\xi_1, \dots, d\xi_n), \Lambda(\xi_1, \dots, \xi_n)) \\
&= \text{Hom}(\Lambda(d\xi_1, \dots, d\xi_n), \Lambda(\xi_1, \dots, \xi_n)) \\
&= \Lambda\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right) \otimes \Lambda(\xi_1, \dots, \xi_n).
\end{aligned} \tag{18}$$

Under the identity

$$x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i \tag{19}$$

we again obtain an isomorphism of chain complexes

$$\Psi : \text{CP}^\bullet(A) \cong \text{CP}^\bullet(A^!).$$

This completes the proof.  $\square$

## 4 Unimodular Poisson algebras and Koszul duality

In this section, we study *unimodular* Poisson algebras/manifolds. As we have stated before, Xu [41] studied this class of Poisson manifolds, and proved the Poincaré duality isomorphism between their Poisson homology and cohomology. Some other related works, inspired by the noncommutative Poincaré duality of Van den Bergh [37], are [22, 27, 43]. The unimodular property of finite dimensional Poisson algebras was studied by Zhu, Van Oystaeyen and Zhang in [42]; in the following we will go over their results in a slightly different way.

### 4.1 Differential calculus of Tamarkin-Tsygan

We first recall the notion of *differential calculus*, introduced by Tamarkin and Tsygan in [33]. Let us start with the definition of Gerstenhaber algebras.

**Definition 4.1.** Recall that a *Gerstenhaber algebra* is the triple  $(V, \wedge, [-, -])$  such that

- (1)  $(V, \wedge)$  is a graded commutative algebra and  $(V, [-, -])$  is a degree 1 or  $-1$  graded Lie algebra;
- (2) the product and Lie bracket are compatible in the following sense

$$[P \wedge Q, R] = [P, R] \wedge Q + (-1)^{p(r-1)} P \wedge [Q, R],$$

for homogeneous  $P, Q, R \in V$  of degree  $p, q, r$ , respectively.

**Definition 4.2** (Differential calculus; Tamarkin-Tsygan [33]). Let  $H^\bullet$  and  $H_\bullet$  be graded vector spaces. A *differential calculus* is the data

$$(H^\bullet, H_\bullet, \wedge, \iota, [-, -], d),$$

satisfying the following conditions:

- (i)  $(H^\bullet, \wedge, [-, -])$  is a Gerstenhaber algebra;

(ii)  $H_\bullet$  is a graded (left) module over  $(H^\bullet, \wedge)$  via the map

$$\iota : H^n \otimes H_m \rightarrow H_{m-n}, \quad f \otimes \alpha \mapsto \iota_f \alpha,$$

for any  $f \in H^n$  and  $\alpha \in H_m$ ;

(iii) There is a map  $d : H_\bullet \rightarrow H_{\bullet+1}$  satisfying  $d^2 = 0$ , and

$$(-1)^{|f|+1} \iota_{[f,g]} = [\mathcal{L}_f, \iota_g] := \mathcal{L}_f \iota_g - (-1)^{|g|(|f|+1)} \iota_g \mathcal{L}_f$$

where  $\mathcal{L}_f := [d, \iota_f] = d\iota_f - (-1)^{|f|} \iota_f d$ .

**Example 4.3** (Calculus on differential manifolds). Suppose  $M$  is an  $n$ -dimensional smooth manifold. Denote by  $\Gamma(\Lambda^\bullet TM)$  the space of poly-vector fields of  $M$ . Under wedge product and the Schouten bracket,  $(\Gamma(\Lambda^\bullet TM), \wedge, [-, -])$  forms a Gerstenhaber algebra. Denote by  $\Omega^\bullet(M)$  the space of differential forms on  $M$ . With the Cartan formula

$$L_X = d \circ \iota_X + \iota_X \circ d, \quad [L_X, \iota_Y] = \iota_{[X,Y]}, \quad (20)$$

where  $d$  is the de Rham differential, we see that

$$(\Gamma(\Lambda^\bullet TM), \Omega^\bullet(M), \wedge, \iota, [-, -], d)$$

forms a differential calculus.

#### 4.1.1 Differential calculus on Poisson (co)homology

Suppose  $A$  is a commutative algebra, and let  $\mathfrak{X}^\bullet(A)$  (in this and only this subsection we drop the subscript  $A$  in  $\mathfrak{X}_A^\bullet(-)$ ) and  $\Omega^\bullet(A)$  be the space of polyvectors and Kähler differentials of  $A$  respectively. We have the following operations on  $\mathfrak{X}^\bullet(A)$  and  $\Omega^\bullet(A)$ :

- (1) Wedge (cup) product: suppose  $P \in \mathfrak{X}^{-p}(A)$  and  $Q \in \mathfrak{X}^{-q}(A)$ , then the *wedge product* of  $P$  and  $Q$ , denoted by  $P \wedge Q$ , is a polyvector in  $\mathfrak{X}^{-p-q}(A)$ , defined by

$$(P \wedge Q)(f_1, f_2, \dots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \dots, f_{\sigma(p)}) \cdot Q(f_{\sigma(p+1)}, \dots, f_{\sigma(p+q)}),$$

where  $\sigma$  runs over all  $(p, q)$ -shuffles of  $(1, 2, \dots, p+q)$ .

- (2) Schouten bracket: suppose  $P \in \mathfrak{X}^{-p}(A)$  and  $Q \in \mathfrak{X}^{-q}(A)$ , then their *Schouten bracket* is an element in  $\mathfrak{X}^{-p-q+1}(A)$ , which is denoted by  $[P, Q]$  and is given by

$$\begin{aligned} [P, Q](f_1, f_2, \dots, f_{p+q-1}) &:= \sum_{\sigma \in S_{q,p-1}} \text{sgn}(\sigma) P(Q(f_{\sigma(1)}, \dots, f_{\sigma(q)}), f_{\sigma(q+1)}, \dots, f_{\sigma(q+p-1)}) \\ &\quad - (-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{p,q-1}} \text{sgn}(\sigma) Q(P(f_{\sigma(1)}, \dots, f_{\sigma(p)}), f_{\sigma(p+1)}, \dots, f_{\sigma(p+q-1)}). \end{aligned}$$

- (3) Contraction (inner product): suppose  $P \in \mathfrak{X}^{-p}(A)$  and  $\omega = df_1 \wedge \cdots \wedge df_n \in \Omega^n(A)$ , then the *contraction* (also called *inner product* or *internal product*) of  $P$  with  $\omega$ , denoted by  $\iota_P(\omega)$ , is an  $A$ -linear map with values in  $\Omega^{n-p}(A)$  given by

$$\iota_P(\omega) = \begin{cases} \sum_{\sigma \in S_{p,n-p}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \dots, f_{\sigma(p)}) df_{\sigma(p+1)} \wedge \cdots \wedge df_{\sigma(n)}, & \text{if } n \geq p, \\ 0, & \text{otherwise.} \end{cases}$$

- (4) Lie derivative: the *Lie derivative* is given by the Cartan formula, namely for  $P \in \mathfrak{X}^{-p}(A)$  and  $\omega \in \Omega^n(A)$ , the Lie derivative of  $\omega$  with respect to  $P$  is given by

$$\mathcal{L}_P \omega := [\iota_P, d] = \iota_P(d\omega) - (-1)^p d(\iota_P \omega),$$

where  $d$  is the de Rham differential.

**Proposition 4.4.** *Suppose  $A$  is a commutative algebra, and let  $\mathfrak{X}^\bullet(A)$  be the space of polyvectors on  $A$ . Then*

- (1)  $(\mathfrak{X}^\bullet(A), \wedge)$  is a graded commutative algebra over  $k$ ;
- (2)  $(\mathfrak{X}^\bullet(A), [-, -])$  is a degree 1 graded Lie algebra over  $k$ ;
- (3) these two structures are compatible in the following sense

$$[P \wedge Q, R] = [P, R] \wedge Q + (-1)^{p(r-1)} P \wedge [Q, R],$$

for all  $P, Q, R \in \mathfrak{X}^\bullet(A)$  of degree  $p, q, r$  respectively.

*Proof.* See [23, Proposition 3.7]. □

Proposition 4.4 exactly says that the space of polyvectors on a commutative algebra forms a Gerstenhaber algebra. Now, the contraction and the Lie derivative satisfy the following proposition:

**Proposition 4.5.** *The contraction  $\iota$  and the Lie derivative  $\mathcal{L}$  satisfy the following property:*

- (1)  $\iota_P \iota_Q = \iota_{Q \wedge P}$ ;
- (2)  $[\mathcal{L}_P, \iota_Q] = \iota_{[P, Q]}$ .

*Proof.* See [23, Propositions 3.4 and 3.11]. □

**Proposition 4.6.** *Suppose  $A$  is a Poisson algebra. Then the Poisson cohomology  $\text{HP}^\bullet(A)$  and homology  $\text{HP}_\bullet(A)$ , together with the induced wedge product, Schouten bracket, contraction, Lie derivative and the de Rham differential, form a differential calculus structure.*

*Proof.* We have to show the operations in Proposition 4.4 and Proposition 4.5 respect the Poisson boundary and coboundary. It is a direct check and can be found in [23, §4.3]. □

In the following, we will give another differential calculus structure for a Poisson algebra, which will be used later:

**Proposition 4.7.** *Suppose  $A$  is a Poisson algebra. Then the pair*

$$(\mathrm{HP}^\bullet(A), \mathrm{HP}^\bullet(A; A^*))$$

*has a differential calculus structure, where  $A^*$  is the dual space of  $A$ .*

*Proof.* Since  $\mathfrak{X}^\bullet(A)$  is already a Gerstenhaber algebra, what remains to show is the existence of the contraction  $\iota : \mathfrak{X}^\bullet(A) \otimes_A \mathfrak{X}^\bullet(A^*) \rightarrow \mathfrak{X}^\bullet(A^*)$ , and a differential on  $\mathfrak{X}^\bullet(A^*)$ , both of which respect the Poisson boundary operators.

(1) For any  $P \in \mathfrak{X}^{-p}(A)$  and  $\phi \in \mathfrak{X}^{-q}(A^*)$ , let  $\iota_P(\phi) \in \mathfrak{X}^{-p-q}(A^*)$  be given by

$$(\iota_P \phi)(f_1, \dots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) P(f_{\sigma(1)}, \dots, f_{\sigma(p)}) \cdot \phi(f_{\sigma(p+1)}, \dots, f_{\sigma(p+q)}).$$

It is clear from Proposition 4.5 (1) that  $\iota$  is associative, i.e.,  $\iota_Q \circ \iota_P = \iota_{P \wedge Q}$ . We next need to show that  $\iota$  respects the Poisson coboundary maps. This is completely analogous to the proof of that  $\wedge$  commutes with the Poisson coboundary map (cf. [23, §4.3]).

(2) Observe that

$$\begin{aligned} \mathfrak{X}^\bullet(A^*) &= \mathrm{Hom}_A(\Omega^\bullet(A), A^*) \\ &= \mathrm{Hom}_A(\Omega^\bullet(A), \mathrm{Hom}(A, k)) \\ &= \mathrm{Hom}_A(\Omega^\bullet(A) \otimes A, k) \\ &= \mathrm{Hom}(\Omega^\bullet(A), k). \end{aligned}$$

By dualizing the de Rham differential  $d$  on  $\Omega^\bullet(A)$ , we obtain a differential  $d^*$  on  $\mathrm{Hom}(\Omega^\bullet(A), k)$ , i.e., on  $\mathfrak{X}^\bullet(A^*)$ . We have to show that  $d^*$  commutes with the Poisson boundary, but this is already proved in [42, Theorem 4.10].  $\square$

In the following, we give two other examples of differential calculus structure on the Hochschild (co)homology of associative algebras. They will be used in later sections.

#### 4.1.2 Calculus on the Hochschild (co)homology, I

Let us first recall the Hochschild theory of associative algebras. Suppose  $A$  is an associative algebra (with unit 1). We denote by  $\bar{A} := A/(k \cdot 1)$  the augmentation ideal of  $A$ , and let  $\pi : A \rightarrow \bar{A}$  be the natural quotient map. Denote  $\bar{a} := \pi(a)$  for any  $a \in A$ . The *reduced (or normalized) bar resolution* of  $A$  is the chain complex  $B^\bullet(A)$  whose  $n$ -th component is the  $k$ -module  $B^n(A) = A \otimes \bar{A}^{\otimes n} \otimes A$ ,  $\forall n \in \mathbb{Z}_{\geq 0}$ . The differential  $\bar{b} : B^n(A) \rightarrow B^{n-1}(A)$  is given by

$$\begin{aligned} \bar{b}(\alpha) &= (a_0 a_1, \bar{a}_2, \dots, \bar{a}_n, a_{n+1}) + \sum_{i=1}^{n-1} (-1)^i (a_0, \bar{a}_1, \dots, \overline{a_i a_{i+1}}, \dots, \bar{a}_n, a_{n+1}) \\ &\quad + (-1)^n (a_0, \bar{a}_1, \dots, \bar{a}_{n-1}, a_n a_{n+1}). \end{aligned}$$

for any  $\alpha = (a_0, \bar{a}_1, \dots, \bar{a}_n, a_{n+1}) \in B^n(A)$ . It is well known that the complex  $(B^\bullet(A), \bar{b})$  is a free resolution of  $A$  as an  $A^e$ -module. Given an  $A$ -bimodule  $M$ , the *(reduced) Hochschild complex* of  $A$  with value in  $M$  is the complex

$$\bar{C}_\bullet(A; M) := M \otimes_{A^e} B^\bullet(A) \cong \bigoplus_{n \geq 0} M \otimes \bar{A}^{\otimes n}$$

with differential  $b : M \otimes \bar{A}^{\otimes n} \rightarrow M \otimes \bar{A}^{\otimes(n-1)}$  given by

$$\begin{aligned} b(\alpha) &= (a_0 a_1, \bar{a}_2, \dots, \bar{a}_n) + \sum_{i=1}^{n-1} (-1)^i (a_0, \bar{a}_1, \dots, \overline{a_i a_{i+1}}, \dots, \bar{a}_n) \\ &\quad + (-1)^n (a_n a_0, \bar{a}_1, \dots, \bar{a}_{n-1}). \end{aligned}$$

for any  $\alpha = (a_0, \bar{a}_1, \dots, \bar{a}_n) \in M \otimes \bar{A}^{\otimes n}$ . The homology of the reduced Hochschild chain complex is the *Hochschild homology* of  $A$  with value in  $M$ , and is denoted by  $\mathrm{HH}_\bullet(A; M)$ . If  $M = A$ , we denote  $\mathrm{HH}_\bullet(A) := \mathrm{HH}_\bullet(A; A)$  the *Hochschild homology* of  $A$ . The *(reduced) Hochschild cochain complex* of  $A$  with value in  $M$  is the complex

$$\bar{C}^\bullet(A; M) := \mathrm{Hom}_{A^e}(\mathrm{B}^\bullet(A), M) \cong \bigoplus_{n \geq 0} \mathrm{Hom}(\bar{A}^{\otimes n}, M)$$

with differential  $\delta : \bar{C}^n(A; M) \rightarrow \bar{C}^{n+1}(A; M)$  given by

$$\begin{aligned} (\delta f)(\bar{a}_1, \dots, \bar{a}_{n+1}) &= a_1 f(\bar{a}_2, \dots, \bar{a}_{n+1}) + \sum_{i=1}^n (-1)^i f(\bar{a}_1, \dots, \overline{a_i a_{i+1}}, \dots, \bar{a}_{n+1}) \\ &\quad + (-1)^{n+1} f(\bar{a}_1, \dots, \bar{a}_n) a_{n+1}, \end{aligned}$$

for any  $f \in \bar{C}^n(A; M)$  and  $(\bar{a}_1, \dots, \bar{a}_{n+1}) \in \bar{A}^{\otimes(n+1)}$ . The homology of the reduced Hochschild cochain complex is the *Hochschild cohomology* of  $A$  with value in  $M$ , and is denoted by  $\mathrm{HH}^\bullet(A; M)$ . If  $M = A$ , we denote  $\mathrm{HH}^\bullet(A) := \mathrm{HH}^\bullet(A; A)$  the *Hochschild cohomology* of  $A$ .

Now we recall the Gerstenhaber algebra structure on the Hochschild cohomology of  $A$ .

- (1) The *Gerstenhaber cup product* on  $\bar{C}^\bullet(A; A)$  is given as follows: for any  $f \in \bar{C}^n(A; A)$  and  $g \in \bar{C}^m(A; A)$ ,

$$f \cup g(\bar{a}_1, \dots, \bar{a}_{n+m}) := (-1)^{nm} f(\bar{a}_1, \dots, \bar{a}_n) g(\bar{a}_{n+1}, \dots, \bar{a}_{n+m}),$$

where  $(\bar{a}_{n+1}, \dots, \bar{a}_{n+m}) \in \bar{A}^{\otimes(n+m)}$ .

- (2) For any  $f \in \bar{C}^n(A; A)$  and  $g \in \bar{C}^m(A; A)$ , let

$$f \circ g(\bar{a}_1, \dots, \bar{a}_{n+m-1}) := \sum_{i=0}^{n-1} (-1)^{(|g|+1)i} f(\bar{a}_1, \dots, \bar{a}_i, \overline{g(\bar{a}_{i+1}, \dots, \bar{a}_{i+m})}, \bar{a}_{i+m+1}, \dots, \bar{a}_{n+m-1});$$

Then the *Gerstenhaber bracket* on  $\bar{C}^\bullet(A; A)$  is defined to be

$$\{f, g\} := f \circ g - (-1)^{(|f|+1)(|g|+1)} g \circ f.$$

It is well-known that the above cup product and bracket are well-defined on homology level and we have the following.

**Theorem 4.8** (Gerstenhaber [14], Theorems 3-5). *Let  $A$  be an associative algebra. Then the Hochschild cohomology  $\mathrm{HH}^\bullet(A)$  of  $A$  equipped with the Gerstenhaber cup product and the Gerstenhaber bracket forms a Gerstenhaber algebra.*

Next, we consider the action of the Hochschild cochain complex on the Hochschild chain complex of  $A$ . Given any homogeneous elements  $f \in \bar{C}^n(A; A)$  and  $\alpha = (a_0, \bar{a}_1, \dots, \bar{a}_m) \in \bar{C}_m(A; A)$ ,

- (1) the *cap product*  $\cap : \bar{C}^n(A; A) \times \bar{C}_m(A; A) \rightarrow \bar{C}_{m-n}(A; A)$  is given by

$$f \cap \alpha := (a_0 f(\bar{a}_1, \dots, \bar{a}_n), \bar{a}_{n+1}, \dots, \bar{a}_m),$$

for  $m \geq n$ , and zero otherwise. If we denote by  $\iota_f(-) := f \cap -$  the contraction operator, then  $\iota_f \iota_g = (-1)^{|f||g|} \iota_{g \cup f} = \iota_{f \cup g}$ ;

- (2) the *Lie derivative*  $L : \bar{C}^n(A; A) \times \bar{C}_m(A; A) \rightarrow \bar{C}_{m-n}(A; A)$  is given as follows: for any  $\alpha = (a_0, \bar{a}_1, \dots, \bar{a}_m) \in \bar{C}_m(A; A)$ , if  $n \leq m+1$ , then

$$\begin{aligned} L_f(\alpha) &:= \sum_{i=0}^{m-n} (-1)^{(n+1)i} (a_0, \bar{a}_1, \dots, \bar{a}_i, \overline{f(\bar{a}_{i+1}, \dots, \bar{a}_{i+n})}, \dots, \bar{a}_m) \\ &+ \sum_{i=m-n+1}^m (-1)^{m(i+1)+n+1} (f(\bar{a}_{i+1}, \dots, \bar{a}_m, \bar{a}_0, \dots, \bar{a}_{n-m+i-1}), \bar{a}_{n-m+i}, \dots, \bar{a}_i), \end{aligned}$$

where the second sum is taken over all cyclic permutations such that  $a_0$  is inside of  $f$ , and otherwise if  $n > m+1$ ,  $L_f(\alpha) = 0$ . In particular,  $L_\mu = -b$ ;

- (3) the *Connes operator*  $B : \bar{C}_\bullet(A; A) \rightarrow \bar{C}_{\bullet+1}(A; A)$  is given by

$$B(\alpha) := \sum_{i=0}^m (-1)^{mi} (1, \bar{a}_i, \dots, \bar{a}_m, \bar{a}_0, \dots, \bar{a}_{i-1}).$$

The following two lemmas first appeared in Daletskii-Gelfand-Tsygan [7], and have been used, for example, by Tamarkin-Tsygan in [33].

**Lemma 4.9.** *Keep the notations as in the above definition. Then*

- (1)  $(\bar{C}_\bullet(A; A), b, \cap)$  is a DG module over  $(\bar{C}^\bullet(A; A), \delta, \cup)$ , that is,

$$\iota_\delta f = (-1)^{|f|+1} [b, \iota_f], \quad \iota_f \iota_g = \iota_{f \cup g}$$

for any homogeneous elements  $f, g \in \bar{C}^\bullet(A; A)$ ;

- (2) for any homogeneous elements  $f, g \in \bar{C}^\bullet(A; A)$ ,

$$[L_f, L_g] = L_{\{f, g\}},$$

and in particular  $(-1)^{|f|+1} [b, L_f] + L_\delta f = 0$ .

**Lemma 4.10** (Homotopy Cartan formulae; [7, 33]). *Suppose  $\iota, L, B$  are given as above and  $f, g \in \bar{C}^\bullet(A; A)$  are any homogeneous elements.*



(1) Define an operation (cf. [33, Equ. (3.5)])

$$S_f(\alpha) := \sum_{i=0}^{m-n} \sum_{j=i+n}^m (-1)^{\eta_{ij}} (1, \bar{a}_{j+1}, \dots, \bar{a}_m, \bar{a}_0, \dots, \bar{a}_i, \overline{f(\bar{a}_{i+1}, \dots, \bar{a}_{i+n})}, \bar{a}_{i+n+1}, \dots, \bar{a}_j)$$

for any  $\alpha = (a_0, \bar{a}_1, \dots, \bar{a}_m) \in \bar{C}_m(A; A)$  (the sum is taken over all cyclic permutations and  $a_0$  always appears on the left of  $f$ ), where  $\eta_{ij} := (n+1)m + (m-j)m + (n+1)(j-i)$ . Then we have

$$L_f = [B, \iota_f] + [b, S_f] - S_{\delta f}. \quad (21)$$

(2) Define

$$T(f, g)(\alpha) := \sum_{i=l-n+2}^l \sum_{j=0}^{n+i-l-2} (-1)^{\theta_{ij}} (f(\bar{a}_{i+1}, \dots, \bar{a}_l, \bar{a}_0, \dots, \bar{a}_j, \overline{g(\bar{a}_{j+1}, \dots, \bar{a}_{j+m})}, \dots, \bar{a}_{n+m+i-l-2}), \dots, \bar{a}_i)$$

for any  $\alpha = (a_0, \bar{a}_1, \dots, \bar{a}_l) \in \bar{C}_l(A; A)$ , where  $\theta_{ij} = (m+1)(i+j+l) + l(i+1)$ . Then we have

$$[L_f, \iota_g] - (-1)^{|f|+1} \iota_{\{f, g\}} = [b, T(f, g)] - T(\delta f, g) - T(f, \delta g). \quad (22)$$

**Theorem 4.11** (Daletskii-Gelfand-Tsygan [7]). *Let  $A$  be an associative algebra. Then the following data of Hochschild (co)homology of  $A$ ,*

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A), \cup, \cap, \{-, -\}, B),$$

*is a differential calculus, where  $\cup$  is the Gerstenhaber cup product,  $\cap$  is the cap product,  $\{-, -\}$  is the Gerstenhaber Lie bracket and  $B$  is the Connes cyclic operator.*

*Proof.* Lemma 4.10 shows that the Cartan formula holds up to homotopy on the chain level, from which and Theorem 4.8 one deduces the desired result.  $\square$

#### 4.1.3 Calculus on the Hochschild (co)homology, II

For an associative algebra  $A$ , denote  $A^* := \mathrm{Hom}(A, k)$ , and then  $A^*$  is an  $A$ -bimodule. Under the identity

$$\bar{C}^\bullet(A; A^*) = \bigoplus_{n \geq 0} \mathrm{Hom}(\bar{A}^{\otimes n}, A^*) = \bigoplus_{n \geq 0} \mathrm{Hom}(A \otimes \bar{A}^{\otimes n}, k),$$

one may equip on  $\bar{C}^\bullet(A; A^*)$  the dual Connes differential, which is denoted by  $B^*$ , i.e.,  $B^*(g) := (-1)^{|g|} g \circ B$  for homogeneous  $g \in \bar{C}^\bullet(A; A^*)$ , and it is well-defined on the homology level. One has the following.

**Theorem 4.12.** *Let  $A$  be an associative algebra. Then*

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}^\bullet(A; A^*), \cup, \cap^*, \{-, -\}, B^*)$$

*is a differential calculus, where the cap product  $\cap^*$  is given by*

$$\begin{aligned} \mathrm{HH}^\bullet(A) \times \mathrm{HH}^\bullet(A; A^*) &\xrightarrow{\cap^*} \mathrm{HH}^\bullet(A; A^*) \\ (f, \alpha) &\longmapsto \iota_f^*(\alpha) := (-1)^{|f||\alpha|} \alpha \circ \iota_f, \end{aligned}$$

*for any homogeneous  $f \in \mathrm{HH}^\bullet(A; A)$  and  $g \in \mathrm{HH}^\bullet(A; A^*)$ .*

*Proof.* By Theorem 4.11, we know that  $(\mathrm{HH}^\bullet(A), \mathrm{HH}^\bullet(A), \cup, \cap, \{-, -\}, B)$  is a differential calculus, and then this theorem holds by the following computations:

(1) By the definition of  $\iota^*$  and Lemma 4.9 (1), one has

$$\begin{aligned}\iota_f^* \iota_g^*(\alpha) &= (-1)^{|g||\alpha|} \iota_f^*(\alpha \circ \iota_g) = (-1)^{|g||\alpha|+|f|(|\alpha|+|g|)} (\alpha \circ \iota_g) \circ \iota_f \\ &= (-1)^{|g||\alpha|+|f|(|\alpha|+|g|)} \alpha \circ (\iota_g \cup f) = (-1)^{|f||g|} \iota_{g \cup f}^* \alpha = \iota_{f \cup g}^*(\alpha),\end{aligned}$$

for any homogenous elements  $f, g \in \mathrm{HH}^\bullet(A)$  and  $\alpha \in \mathrm{HH}^\bullet(A; A^*)$ . This means that the cap product is a left module.

(2) Given any homogenous elements  $f \in \mathrm{HH}^\bullet(A)$  and  $\alpha \in \mathrm{HH}^\bullet(A; A^*)$ , we define  $L_f^*(\alpha) := (-1)^{|f||\alpha|+|\alpha|+1} \alpha \circ L_f (= [B^*, \iota_f^*](\alpha))$ , and by Lemma 4.10 one has

$$\begin{aligned}[L_f^*, \iota_g^*](\alpha) &= (L_f^* \iota_g^* - (-1)^{(|f|+1)|g|} \iota_g^* L_f^*)(\alpha) \\ &= (-1)^{(|f|+1)(|\alpha|+|g|)+|g||\alpha|+1} \alpha \circ (\iota_g L_f) - (-1)^{(|f|+|g|+1)|\alpha|+1} \alpha \circ (L_f \iota_g) \\ &= (-1)^{(|f|+|g|+1)|\alpha|} \alpha \circ ([L_f, \iota_g]) \\ &= (-1)^{(|f|+|g|+1)|\alpha|} \alpha \circ ((-1)^{|f|+1} \iota_{\{f, g\}}) \\ &= (-1)^{|f|+1} \iota_{\{f, g\}}^*(\alpha).\end{aligned}$$

This completes the proof.  $\square$

## 4.2 Unimodular Poisson manifolds

Unimodular Poisson algebras arises in Poisson geometry. Suppose  $M$  is an  $n$ -dimensional oriented Poisson manifold. Let  $A$  be the algebra of smooth functions on  $M$ . We have

$$\mathfrak{X}_A^\bullet(A) = \mathfrak{X}^\bullet(M), \quad \Omega^\bullet(A) = \Omega^\bullet(M),$$

where  $\mathfrak{X}^\bullet(M)$  and  $\Omega^\bullet(M)$  are the spaces of polyvectors and differential forms on  $M$  respectively. Let  $\eta$  be a nowhere vanishing  $n$ -form on  $M$ . We have the following diagram

$$\begin{array}{ccc} \mathfrak{X}_A^\bullet(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n+\bullet}(A) \\ \uparrow \delta & & \uparrow \partial \\ \mathfrak{X}_A^{\bullet+1}(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n+\bullet+1}(A), \end{array} \quad (23)$$

which may not be commutative, i.e.,  $\eta$  may not be a Poisson cycle. There is a well-defined cohomology class in  $\mathrm{HP}^1(M)$ , which does not depend on the choice of  $\eta$ , obstructing (23) to be commutative. It is represented by the divergence of the Poisson bivector, and is called the *modular class* of  $M$ . If the modular class vanishes, then we say  $M$  (or  $A$ ) is *unimodular*.

For more details on unimodular Poisson manifolds, see Weinstein [38], Xu [41], or the monograph [23, §4.4]. From the definition, one immediately obtains:

**Theorem 4.13** (Xu [41]). *Suppose  $M$  is an oriented unimodular Poisson manifold of dimension  $n$ . Then there is an isomorphism*

$$\mathrm{HP}^\bullet(M) \cong \mathrm{HP}_{n-\bullet}(M).$$

*Proof.* See Xu [41, Theorem 4.8]. □

The definition of unimodularity has a purely algebraic description, which is given by Dolgushev [9], and has been later studied in, for example, [12, 27, 43]. In this case, the obstructions for (23) being commutative are characterized by the divergence of the Poisson bivector up to *log-Hamiltonian vectors*. We shall work in the sense of Dolgushev, however, we do not need go to the details of that, since all we need is the diagram (23); if it is commutative for some  $\eta$ , then we say the Poisson algebra is *unimodular*. The Poincaré duality for Poisson cohomology and homology in this case was studied by Etingof and Ginzburg (see [12, Proposition 5.1.1]).

### 4.3 Unimodular cyclic Poisson algebras

Now, we go to unimodular cyclic Poisson algebras, a notion introduced by Zhu, Van Oystaeyen and Zhang in [42].

Suppose  $A^!$  is a finite dimensional graded Poisson algebra. If there is an  $A^!$ -bimodule isomorphism

$$\eta^! : (A^!)^\bullet \longrightarrow (A^i)_{n+\bullet}, \quad \text{for some } n \in \mathbb{N},$$

where  $A^i = (A^!)^*$  (note that this is equivalent to saying that  $A^!$  admits a degree  $n$  cyclically invariant non-degenerate pairing; see also §6.2 below), then we may view  $\eta^!$  as an element in  $\text{Hom}_{A^!}(A^!, A^i) \subset \mathfrak{X}_{A^!}^\bullet(A^i)$ , and have a diagram

$$\begin{array}{ccc} \mathfrak{X}_{A^!}^\bullet(A^!) & \xrightarrow{\iota_{(-)}\eta^!} & \mathfrak{X}_{A^!}^{n+\bullet}(A^i) \\ \uparrow \delta & & \uparrow \delta \\ \mathfrak{X}_{A^!}^{\bullet+1}(A^!) & \xrightarrow{\iota_{(-)}\eta^!} & \mathfrak{X}_{A^!}^{n+\bullet+1}(A^i). \end{array} \quad (24)$$

**Definition 4.14** (Unimodular cyclic Poisson algebra; [42]). Suppose  $A^!$  is a finite dimensional (graded) Poisson algebra. If there is an  $\eta^! \in \mathfrak{X}_{A^!}^\bullet(A^i)$  (also called the *volume form*) such that the digram (24) commutes, then  $A^!$  is called a *unimodular cyclic Poisson algebra* of degree  $n$ .

**Theorem 4.15** (Zhu-Van Oystaeyen-Zhang [42]). Suppose  $A^!$  is a unimodular cyclic Poisson algebra with the volume form of degree  $n$ . Then there exists an isomorphism

$$\text{HP}^\bullet(A^!, \pi^!) \cong \text{HP}^{\bullet-n}(A^!, \pi^!; A^i).$$

*Proof.* This is a tautology from the definition. □

### 4.4 Unimodular quadratic Poisson algebras

**Theorem 4.16** (First half of Theorem 1.2). Suppose  $A = k[x_1, \dots, x_n]$  is a quadratic Poisson algebra. Then  $(A, \pi)$  is unimodular if and only if  $(A^!, \pi^!)$  is unimodular cyclic.

*Proof.* We construct two volume forms  $\eta$  and  $\eta^!$ , completing the diagrams (23) and (24), and show that the following diagram

$$\begin{array}{ccc} \mathfrak{X}_A^\bullet(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^\bullet(A) \\ \downarrow & & \downarrow \\ \mathfrak{X}_{A^!}^\bullet(A^!) & \xrightarrow{\iota_{(-)}\eta^!} & \mathfrak{X}_{A^!}^\bullet(A^i) \end{array} \quad (25)$$

under the natural maps given by (19), commutes.

In fact, recall that for  $A = k[x_1, \dots, x_n]$ ,

$$\begin{aligned} \mathfrak{X}_A^\bullet(A) &= \Lambda\left(x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), & \Omega^\bullet(A) &= \Lambda(x_1, \dots, x_n, dx_1, \dots, dx_n), \\ \mathfrak{X}_{A^!}^\bullet(A^!) &= \Lambda\left(\xi_1, \dots, \xi_n, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right), & \mathfrak{X}_{A^!}^\bullet(A^i) &= \Lambda\left(\xi_1^*, \dots, \xi_n^*, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right). \end{aligned}$$

Let

$$\eta = dx_1 dx_2 \cdots dx_n \quad \text{and} \quad \eta^! = \xi_1^* \xi_2^* \cdots \xi_n^*,$$

where  $\eta^!$  is understood as contraction, namely,

$$\eta^!(\xi_{i_1} \cdots \xi_{i_p}) := \sum_{\sigma \in S_{p, n-p}} \langle \xi_{i_1} \cdots \xi_{i_p}, \xi_{\sigma(1)}^* \cdots \xi_{\sigma(p)}^* \rangle \cdot \xi_{\sigma(p+1)}^* \cdots \xi_{\sigma(n)}^*,$$

then under the identification

$$x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad dx_i \mapsto \xi_i^*, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i$$

the diagram

$$\begin{array}{ccc} \mathfrak{X}_A^\bullet(A) = \Lambda\left(x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) & \xrightarrow{\iota_{(-)}dx_1 \cdots dx_n} & \Omega^\bullet(A) = \Lambda(x_1, \dots, x_n, dx_1, \dots, dx_n) \\ \downarrow \cong & & \downarrow \cong \\ \mathfrak{X}_{A^!}^\bullet(A^!) = \Lambda\left(\xi_1, \dots, \xi_n, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right) & \xrightarrow{\iota_{(-)}\xi_1^* \cdots \xi_n^*} & \mathfrak{X}_{A^!}^\bullet(A^i) = \Lambda\left(\xi_1^*, \dots, \xi_n^*, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right) \end{array} \quad (26)$$

commutes. This is exactly (25), which completes the proof.  $\square$

**Theorem 4.17** (Second half of Theorem 1.2). *Suppose  $A = k[x_1, \dots, x_n]$  is a unimodular quadratic Poisson algebra, and let  $A^!$  be its Koszul dual. Then we have the following commutative diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HP}^\bullet(A^!) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^!; A^i) \end{array} \quad (27)$$

*Proof.* By Theorem 4.16,  $A^!$  is unimodular cyclic. Therefore, the two vertical isomorphisms are given by Theorem 3.4, and the two horizontal isomorphisms are given by Theorems 4.13 and 4.15 respectively. Chosing two dual volume forms  $\eta = dx_1 dx_2 \cdots dx_n$  and  $\eta^! = \xi_1^* \xi_2^* \cdots \xi_n^*$ , the commutativity of the diagram (27) follows from the chain level commutative diagram (26).  $\square$

## 5 Poisson cohomology and the Batalin-Vilkovisky algebra

The purpose of this section is to show that for unimodular quadratic Poisson polynomial algebras, the horizontal isomorphisms in (27) naturally induces on  $\mathrm{HP}^\bullet(A)$  and  $\mathrm{HP}^\bullet(A^!)$  a Batalin-Vilkovisky algebra structure, and the vertical isomorphisms in (27) are isomorphisms of Batalin-Vilkovisky algebras.

**Definition 5.1** (Lambre [21]). A differential calculus  $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \wedge, \iota, [-, -], d)$  is called a *differential calculus with duality* if there exists an integer  $n$  and an element  $\eta \in \mathrm{H}_n$  such that

(a)  $\iota_1 \eta = \eta$ , where  $1 \in \mathrm{H}^0$  is the unit,  $d(\eta) = 0$ , and

(b) for any  $i \in \mathbb{Z}$ ,

$$\mathrm{PD}(-) := \iota_{(-)} \eta : \mathrm{H}^i \rightarrow \mathrm{H}_{n-i} \quad (28)$$

is an isomorphism.

Such isomorphism  $\mathrm{PD}$  is called the *Van den Bergh's duality* (also called *the noncommutative Poincaré duality*), and  $\eta$  is called the *volume form*. We denote by  $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \wedge, \iota, [-, -], d, \eta)$  the differential calculus with duality.

**Definition 5.2** (Batalin-Vilkovisky algebra). Suppose  $(V, \bullet)$  is an graded commutative algebra. A *Batalin-Vilkovisky algebra* structure on  $V$  is the triple  $(V, \bullet, \Delta)$  such that

(1)  $\Delta : V^i \rightarrow V^{i-1}$  is a differential, that is,  $\Delta^2 = 0$ ; and

(2)  $\Delta$  is second order operator, that is,

$$\begin{aligned} \Delta(a \bullet b \bullet c) &= \Delta(a \bullet b) \bullet c + (-1)^{|a|} a \bullet \Delta(b \bullet c) + (-1)^{(|a|-1)|b|} b \bullet \Delta(a \bullet c) \\ &\quad - (\Delta a) \bullet b \bullet c - (-1)^{|a|} a \bullet (\Delta b) \bullet c - (-1)^{|a|+|b|} a \bullet b \bullet (\Delta c). \end{aligned}$$

Equivalently, if we define the bracket

$$[a, b] := (-1)^{|a|+1} (\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)),$$

then  $[-, -]$  is a derivation with respect to  $\bullet$  for each component. In other words, a Batalin-Vilkovisky algebra is a Gerstenhaber algebra  $(V, \bullet, [-, -])$  with a differential  $\Delta : V^i \rightarrow V^{i-1}$  such that

$$[a, b] = (-1)^{|a|+1} (\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)), \quad (29)$$

for any  $a, b \in V$  (cf. [15, Proposition 1.2]).

Now suppose  $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \wedge, \iota, [-, -], d, \eta)$  is a differential calculus with duality. Let  $\Delta : \mathrm{H}^\bullet \rightarrow \mathrm{H}^{\bullet-1}$  be the linear operator such that

$$\begin{array}{ccc} \mathrm{H}^\bullet & \xrightarrow{\Delta} & \mathrm{H}^{\bullet-1} \\ \downarrow \mathrm{PD} & & \downarrow \mathrm{PD} \\ \mathrm{H}_{n-\bullet} & \xrightarrow{d} & \mathrm{H}_{n-\bullet+1} \end{array} \quad (30)$$

commutes. Then we have the following theorem due to Lambre (see [21, Théorème 1.6]):

**Theorem 5.3** (Lambre [21]). *Let  $(H^\bullet, H_\bullet, \wedge, \iota, [-, -], d, \eta)$  be a differential calculus with duality. Then the triple  $(H^\bullet, \wedge, \Delta)$  is a Batalin-Vilkovisky algebra.*

*Proof.* Since  $(H^\bullet, \wedge, [-, -])$  is a Gerstenhaber algebra, we only need to show that the Gerstenhaber bracket is compatible with the operator  $\Delta$  in (30); that is, equation (29) holds. For any homogeneous elements  $f, g \in H^\bullet$ , by the definition of Poincaré duality PD (28) and the Cartan formulae (Lemma 4.10), we have

$$\begin{aligned}
& (-1)^{|f|+1} \text{PD}([f, g]) \\
&= (-1)^{|f|+1} \iota_{[f, g]}(\eta) = [\mathcal{L}_f, \iota_g](\eta) = \mathcal{L}_f \iota_g(\eta) - (-1)^{|g|(|f|+1)} \iota_g \mathcal{L}_f(\eta) \\
&= d \iota_f \iota_g(\eta) - (-1)^{|f|} \iota_f d \iota_g(\eta) - (-1)^{|g|(|f|+1)} \iota_g d \iota_f(\eta) + (-1)^{|g|(|f|+1)+|f|} \iota_g \iota_f d(\eta) \\
&= d \circ \text{PD}(f \wedge g) - (-1)^{|g|(|f|+1)} \iota_g d \circ \text{PD}(f) - (-1)^{|f|} \iota_f d \circ \text{PD}(g) \\
&= \text{PD}(\Delta(f \wedge g)) - (-1)^{|g|(|f|+1)} \iota_g \text{PD}(\Delta(f)) - (-1)^{|f|} \iota_f \text{PD}(\Delta(g)) \\
&= \iota_{\Delta(f \wedge g)}(\eta) - (-1)^{|g|(|f|+1)} \iota_g \iota_{\Delta(f)}(\eta) - (-1)^{|f|} \iota_f \iota_{\Delta(g)}(\eta) \\
&= (\iota_{\Delta(f \wedge g)} - (-1)^{|g|(|f|+1)} \iota_g \iota_{\Delta(f)} - (-1)^{|f|} \iota_f \iota_{\Delta(g)})(\eta) \\
&= \text{PD}(\Delta(f \wedge g) - \Delta(f) \wedge g - (-1)^{|f|} f \wedge \Delta(g)).
\end{aligned}$$

Since PD is an isomorphism, we thus have

$$[f, g] = (-1)^{|f|+1} (\Delta(f \wedge g) - \Delta(f) \wedge g - (-1)^{|f|} f \wedge \Delta(g)). \quad \square$$

**Corollary 5.4** (Xu [41] and Zhu-Van Oystaeyen-Zhang [42]). *Suppose  $A$  is a unimodular Poisson or unimodular cyclic Poisson algebra. Then  $\text{HP}^\bullet(A)$  admits a Batalin-Vilkovisky algebra structure.*

*Proof.* If  $A$  is unimodular Poisson, then the statement follows from Proposition 4.6, Theorems 4.13 and Theorem 5.3. If  $A$  is unimodular cyclic Poisson, then the statement follows from Proposition 4.7, Theorem 4.15 and Theorem 5.3.  $\square$

**Theorem 5.5** (Theorem 1.3). *Suppose  $A = k[x_1, \dots, x_n]$  is a unimodular quadratic Poisson algebra, and let  $A^!$  be its Koszul dual. Then there is an isomorphism*

$$\text{HP}^\bullet(A) \cong \text{HP}^\bullet(A^!)$$

*of Batalin-Vilkovisky algebras.*

*Proof.* Note that in Theorem 4.17, the right vertical isomorphism preserves the de Rham differential as well as the volume form, that is, the two differential calculus with duality

$$(\text{HP}^\bullet(A), \text{HP}_\bullet(A)) \text{ and } (\text{HP}^\bullet(A^!), \text{HP}^\bullet(A^!; A^i))$$

are isomorphic. Combining with Corollary 5.4, the theorem follows.  $\square$

**Remark 5.6.** Not all quadratic Poisson algebras are unimodular. For example, for  $A = \mathbb{C}[x_1, x_2, x_3]$ , Etingof-Ginzburg [12, Lemma 4.2.3 and Corollary 4.3.2] showed that any unimodular Poisson structure is of the form

$$\{x, y\} = \frac{\partial \phi}{\partial z}, \quad \{y, z\} = \frac{\partial \phi}{\partial x}, \quad \{z, x\} = \frac{\partial \phi}{\partial y},$$

for some  $\phi \in A$  (taking  $\phi$  to be cubic then the Poisson structure is quadratic); for  $A = \mathbb{C}[x_1, x_2, x_3, x_4]$ , Pym [30, §3] showed that any unimodular quadratic Poisson bracket on  $A$  may be written uniquely in the following form

$$\{f, g\} := \frac{df \wedge dg \wedge d\alpha}{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}, \quad f, g \in A,$$

where  $\alpha = \sum_{i=1}^4 \alpha_i dx_i \in \Omega^1(A)$  such that  $\alpha \wedge d\alpha = 0$ , and  $\alpha_i$ 's are homogeneous cubic polynomials satisfying  $\sum_{i=1}^4 x_i \alpha_i = 0$ .

## 6 Calabi-Yau algebras

At the end of §1 we sketched some analogy between unimodular Poisson algebras and Calabi-Yau algebras. In this and the following section, we study their relationships in more detail.

### 6.1 Calabi-Yau algebras and the Batalin-Vilkovisky algebra structure

Let us first remind the definition of Calabi-Yau algebras (see Ginzburg [16]):

**Definition 6.1** (Calabi-Yau algebra; Ginzburg [16]). Let  $A$  be an associative algebra over  $k$ .  $A$  is called a *Calabi-Yau algebra of dimension  $n$*  if

- (1)  $A$  is homologically smooth, that is,  $A$ , viewed as an  $A^e$ -module, has a finite resolution of finitely generated projective  $A^e$ -modules, and
- (2) there is an isomorphism

$$\mathrm{RHom}_{A^e}(A, A \otimes A) \cong \Sigma^{-n} A \quad (31)$$

in the derived category  $D(A^e)$  of  $A^e$ -modules.

In the above definition,  $A^e$  is the enveloping algebra of  $A$ , namely  $A^e := A \otimes A^{\mathrm{op}}$ . Calabi-Yau algebras are noncommutative generalizations of Calabi-Yau varieties in the sense that, suppose  $A$  is a coordinate ring of an affine variety  $X$  of dimension  $n$ , then  $A$  is  $n$ -Calabi-Yau if and only if  $X$  is Calabi-Yau (that is,  $X$  is smooth with trivial canonical sheaf) (cf. [12, Example 1.4.3]). There are a lot of other examples of Calabi-Yau algebras, such as the universal enveloping algebra of semi-simple Lie algebras, the skew-product of complex polynomials with a finite subgroup of  $\mathrm{SL}(n, \mathbb{C})$ , the Yang-Mills algebras, etc.

One of the most importance is the following result, due to Van den Bergh [37], which gives the “noncommutative” Poincaré duality of Calabi-Yau algebras.

**Theorem 6.2** (Van den Bergh [37]). *Suppose  $A$  is a Calabi-Yau algebra of dimension  $n$ . Then there exists a class  $\eta \in \mathrm{HH}_n(A)$  such that the contraction*

$$\mathrm{HH}^\bullet(A) \xrightarrow{-\cap \eta} \mathrm{HH}_{n-\bullet}(A)$$

*is an isomorphism, where  $\cap$  is the cap product (contraction) of the Hochschild cohomology on Hochschild homology. In particular,  $(\mathrm{HH}^\bullet(A), \cup, \Delta)$  forms a Batalin-Vilkovisky algebra.*

For a complete proof, see [8, Proposition 5.5]. In order to compare with the differential calculus with duality for unimodular Poisson algebras, we give a proof of this theorem in the following.

To this end, let us first recall that, for an associative algebra  $A$ ,  $\bar{C}^\bullet(A; A) \cong \text{RHom}_{A^e}(A, A)$  and  $\bar{C}_\bullet(A; A) \cong A \otimes_{A^e}^L A$  (cf. Loday [25]). Recall that  $\text{RHom}_{A^e}(A, A)$  is equipped with the Yoneda cup product, which acts on  $A \otimes_{A^e}^L A$  naturally from the right. More generally, suppose  $M$  is an  $A^e$ -module; since  $\text{RHom}_{A^e}(M, A) \cong \text{RHom}_{A^e}(M, B^\bullet(A))$ , one has the following (Yoneda cup and cap) maps

$$\begin{array}{ccc} \text{RHom}_{A^e}(M, B^\bullet(A)) \otimes \text{Hom}_{A^e}(B^\bullet(A), B^\bullet(A)) & \xrightarrow{\circ} & \text{RHom}_{A^e}(M, B^\bullet(A)) \\ (f, & g) & \mapsto g \circ f \end{array} \quad (32)$$

and

$$\begin{array}{ccc} (M \otimes_{A^e} B^\bullet(A)) \otimes \text{Hom}_{A^e}(B^\bullet(A), B^\bullet(A)) & \longrightarrow & M \otimes_{A^e} B^\bullet(A) \\ ((m_0, a_1, \dots, a_l), & f) & \mapsto (m_0, f(a_1, \dots, a_l)) \end{array} \quad (33)$$

The following statement is well-known in homological algebra since Gerstenhaber:

**Lemma 6.3.** *Let  $A$  be an algebra and  $M$  an  $A^e$  module. Then with (32) and (33) given above,*

$$\text{RHom}_{A^e}(M, B^\bullet(A)) \quad \text{and} \quad M \otimes_{A^e} B^\bullet(A)$$

*become DG modules over  $\text{Hom}_{A^e}(B^\bullet(A), B^\bullet(A))$ . In particular, if  $M = A$ , then the actions of  $\text{Hom}_{A^e}(B^\bullet(A), B^\bullet(A))$  given by (32) and (33), when passing to the (co)homology level, correspond to the Gerstenhaber cup and cap products respectively.*

*Proof of Theorem 6.2.* Since  $A$  is homologically smooth and  $\text{Ext}_{A^e}^j(A, A^e) = 0$  for  $j \neq n$ , there exists a projective resolution of  $A$  as  $A^e$ -modules:

$$0 \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \dots \longrightarrow P^0 \longrightarrow A \longrightarrow 0,$$

where each  $P^i$  finitely generated projective. Denote by  $B^\bullet(A)$  the bar resolution of  $A$ :

$$\dots \longrightarrow A \otimes \bar{A} \otimes A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0,$$

then

$$\begin{aligned} \text{RHom}_{A^e}(A, A) &= \text{Hom}_{A^e}(P^\bullet, B^\bullet(A)) \\ &= \text{Hom}_{A^e}(P^\bullet, A^e) \otimes_{A^e} B^\bullet(A) \\ &\cong \Sigma^{-n} A \otimes_{A^e} B^\bullet(A) \end{aligned} \quad (34)$$

in  $\mathcal{D}(k)$ , where the third equality holds due to equation (31). Denote the above isomorphism (34) by  $\phi$ , then by Lemma 6.3, the following diagram

$$\begin{array}{ccc} \left( \text{Hom}_{A^e}(P^\bullet, B^\bullet(A)) \right) \otimes \left( \text{Hom}_{A^e}(B^\bullet(A), B^\bullet(A)) \right) & \xrightarrow{\cup} & \text{Hom}_{A^e}(P^\bullet, B^\bullet(A)) \\ \downarrow \phi \times id & & \downarrow \phi \\ \left( \Sigma^{-n} A \otimes_{A^e} B^\bullet(A) \right) \otimes \left( \text{Hom}_{A^e}(B^\bullet(A), B^\bullet(A)) \right) & \xrightarrow{\cap} & \Sigma^{-n} A \otimes_{A^e} B^\bullet(A) \end{array}$$



is commutative. Thus, on the homology level, we have that

$$\phi : \mathrm{HH}^\bullet(A) \cong \mathrm{HH}_{n-\bullet}(A)$$

is an isomorphism of  $\mathrm{HH}^\bullet(A)$  modules. Let  $[id] \in \mathrm{HH}^0(A)$ , and  $\phi([id]) = \eta \in \mathrm{HH}_n(A)$ , then for any  $f \in \mathrm{HH}^\bullet(A)$ ,

$$\phi(f) = \phi([id] \cup f) = \phi([id]) \cap f = \eta \cap f,$$

which means  $\eta$  is a volume form, and hence by Theorem 4.11,

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A), \cup, [-, -], \cap, B, \eta)$$

is a differential calculus with duality. This proves the theorem.  $\square$

*Proof of Lemma 6.3.* Formulas (32) and (33) respects the differential, and since the compositions of morphisms are associative, the first half of the lemma is clear.

For the second half, suppose  $[f] \in \mathrm{HH}^l(A)$  and  $[g] \in \mathrm{HH}^m(A)$  are represented by  $f : \bar{A}^{\otimes l} \rightarrow A$  and  $g : \bar{A}^{\otimes m} \rightarrow A$  respectively. Under the the isomorphism

$$\phi : \mathrm{Hom}_k(\bar{A}^{\otimes l}, A) \cong \mathrm{Hom}_{A^e}(A \otimes \bar{A}^{\otimes l} \otimes A, A)$$

$f$  is mapped to  $\phi(f)$ , which maps  $(1, \bar{a}_1, \dots, \bar{a}_l, 1)$  to  $f(a_1, \dots, a_l)$ , and extends to  $A \otimes \bar{A}^{\otimes l} \otimes A$  by  $A^e$  linearity. We thus have

$$\begin{aligned} \phi(f \cup g)(a_0, \bar{a}_1, \dots, \bar{a}_{m+l}, a_{m+l+1}) &= a_0 f \cup g(\bar{a}_1, \dots, \bar{a}_{m+l}) a_{m+l+1} \\ &= a_0 f(\bar{a}_1, \dots, \bar{a}_l) g(\bar{a}_{l+1}, \dots, \bar{a}_{m+l}) a_{m+l+1}. \end{aligned} \quad (35)$$

On the other hand, viewing  $\phi(f) \in \mathrm{Hom}_{A^e}(A \otimes \bar{A}^{\otimes l} \otimes A, A) = \mathrm{RHom}_{A^e}^l(\mathrm{B}^\bullet(A), A)$ ,  $\phi(f)$  has a lifting  $\widetilde{\phi(f)} \in \mathrm{Hom}_{A^e}^l(\mathrm{B}^\bullet(A), \mathrm{B}^\bullet(A))$  via the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & A \otimes \bar{A}^{l+1} \otimes A & \longrightarrow & A \otimes \bar{A}^l \otimes A & \longrightarrow & \dots \longrightarrow A \otimes A \\ & & \downarrow \widetilde{\phi(f)} & & \downarrow \widetilde{\phi(f)} & \searrow \phi(f) & \\ \dots & \longrightarrow & A \otimes \bar{A} \otimes A & \longrightarrow & A \otimes A & \longrightarrow & A \end{array}$$

which is given by

$$\widetilde{\phi(f)}(a_0, \bar{a}_1, \dots, \bar{a}_i, a_{i+1}) = \begin{cases} (a_0 f(\bar{a}_1, \dots, \bar{a}_l), \bar{a}_{l+1}, \dots, \bar{a}_i, a_{i+1}), & \text{if } i \geq l, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the Yoneda product  $[\phi(g)] \circ [\phi(f)]$  of  $[\phi(f)]$  and  $[\phi(g)]$  is represented by

$$\begin{aligned} \phi(g) \circ \widetilde{\phi(f)}(a_0, \bar{a}_1, \dots, \bar{a}_{m+l}, a_{m+l+1}) &= \phi(g)(a_0 f(\bar{a}_1, \dots, \bar{a}_l), \bar{a}_{l+1}, \dots, \bar{a}_{m+l}, a_{m+l+1}) \\ &= (a_0 f(\bar{a}_1, \dots, \bar{a}_l) g(\bar{a}_{l+1}, \dots, \bar{a}_{m+l}) a_{m+l+1}) \\ &\stackrel{(35)}{=} \phi(f \cup g)(a_0, \bar{a}_1, \dots, \bar{a}_{m+l}, a_{m+l+1}). \end{aligned}$$

That is,  $[\phi(g)] \circ [\phi(f)] = [\phi(f \cup g)]$ .

The proof of the Yoneda cap product being compatible with the Gerstenhaber cap product is similar, and is left to the interested reader.  $\square$

## 6.2 Cyclic algebras and the Batalin-Vilkovisky algebra structure

Recall that an associative algebra (possibly graded), say  $A^!$ , is called *cyclic* if it is equipped with a bilinear, non-degenerate symmetric pairing

$$\langle -, - \rangle : A^! \otimes A^! \rightarrow k$$

of degree  $n$  satisfying the cyclically invariant  $\langle a, b \cdot c \rangle = (-1)^{(|a|+|b|)|c|} \langle c, a \cdot b \rangle$ , for all homogeneous  $a, b, c \in A^!$ .

**Theorem 6.4** (Tradler [34]). *Suppose  $A^!$  is a cyclic algebra of degree  $n$ . Then there exists a class  $\eta \in \mathrm{HH}^{-n}(A^!; A^i)$  such that*

$$\mathrm{HH}^\bullet(A^!) \xrightarrow{-\cap^* \eta} \mathrm{HH}^{\bullet-n}(A^!; A^i)$$

*is an isomorphism, where  $\cap^*$  is the cap product as in Theorem 4.12. In particular,  $\mathrm{HH}^\bullet(A^!)$  is a Batalin-Vilkovisky algebra.*

*Proof.* As in the proof of Theorem 6.2, we only need to show the existence of the volume form. Recall that the existence of the degree  $n$  cyclic pairing is equivalent to an isomorphism

$$\eta : A^! \cong \Sigma^{-n} A^i$$

as  $A^!$ -bimodules. Such  $\eta$  may be viewed as an element in  $\bar{C}^{-n}(A^!; A^i)$ , which is a cocycle, and hence represents a cohomology class. By abuse of notation, this class is also denoted by  $\eta$ . The following map

$$-\cap^* \eta : \bar{C}^\bullet(A^!; A^i) = \bigoplus_{q \geq 0} \mathrm{Hom}((\bar{A}^!)^{\otimes q}, A^i) \xrightarrow{\eta \circ -} \bigoplus_{q \geq 0} \mathrm{Hom}((\bar{A}^!)^{\otimes q}, \Sigma^{-n} A^i) = \bar{C}^{\bullet-n}(A^!; A^i),$$

where  $\eta \circ -$  means composing with  $\eta$ , gives the desired isomorphism on the cohomology. Therefore, the last claim follows from Theorems 4.12 and 5.3.  $\square$

## 6.3 Koszul Calabi-Yau algebras and Rouquier's conjecture

Analogous to the quadratic Poisson algebra case, the Koszul dual of a Koszul Calabi-Yau algebra is cyclic (chronologically the latter is discovered first):

**Theorem 6.5.** *Suppose  $A$  is a Koszul algebra and let  $A^!$  be its Koszul dual algebra. Then  $A$  is Calabi-Yau of dimension  $n$  if and only if its Koszul dual  $A^!$  is cyclic of degree  $n$ .*

*Proof.* See, for example, Van den Bergh [36, Theorem 9.2], or [6, Proposition 28].  $\square$

Thus for a Koszul Calabi-Yau algebra  $A$ , by Theorems 4.11, 6.2 and 6.4, we have two differential calculus with duality:

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A)) \quad \text{and} \quad (\mathrm{HH}^\bullet(A^!), \mathrm{HH}^\bullet(A^!; A^i)),$$

and hence two Batalin-Vilkovisky algebras. It has been well-known that for a Koszul algebra, say  $A$ ,

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(A^!),$$

as Gerstenhaber algebras, and Rouquier conjectured (it is stated in Ginzburg [16]) that, for a Koszul Calabi-Yau algebra, the above two Batalin-Vilkovisky are isomorphic.

**Theorem 6.6** (Rouquier's conjecture). *Suppose  $A$  is a Koszul Calabi-Yau algebra of dimension  $n$ , and denote by  $A^!$  its Koszul dual. Then*

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(A^!)$$

*as Batalin-Vilkovisky algebras.*

*Proof.* See [6, Theorem A]. □

Let us give some remarks of the proof, which supplement some details of [6]. In fact, the proof is completely analogous to the Poisson case given in the previous sections (Theorem 5.5), and is divided into three steps:

- (1) by means of Koszul duality, the complex  $A \otimes A^!$  (respectively  $A \otimes A^i$ ), with differentials properly assigned, computes both  $\mathrm{HH}^\bullet(A)$  and  $\mathrm{HH}^\bullet(A^!)$  (respectively  $\mathrm{HH}_\bullet(A)$  and  $\mathrm{HH}^\bullet(A^!; A^i)$ );
- (2) one can also show that in the isomorphism

$$\mathrm{HH}_\bullet(A) \cong \mathrm{H}_\bullet(A \otimes A^i) \cong \mathrm{HH}^\bullet(A^!; A^i),$$

the Connes cyclic operator on  $\mathrm{HH}_\bullet(A)$  is mapped to the dual Connes cyclic operator on  $\mathrm{HH}^\bullet(A^!; A^i)$ , and hence

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A)) \quad \text{and} \quad (\mathrm{HH}^\bullet(A^!), \mathrm{HH}_\bullet(A^!; A^i)) \tag{36}$$

are isomorphic as differential calculus;

- (3) again via the isomorphism

$$\mathrm{HH}_\bullet(A) \cong \mathrm{H}_\bullet(A \otimes A^i) \cong \mathrm{HH}^\bullet(A^!; A^i),$$

one sees that the volume form in  $\mathrm{HH}_\bullet(A)$  is mapped to the volume form in  $\mathrm{HH}^\bullet(A^!; A^i)$  (more precisely it is represented by non-zero element in  $k \otimes A_n^i \subset A \otimes A^i$ ).

Therefore, we obtain that the two differential calculus structures (36) are in fact isomorphic as differential calculus with duality, and the theorem follows.

**Remark 6.7.** We have to mention that in [17] Herscovich has independently obtained that for Koszul algebras, the two pairs in (36) are isomorphic as differential calculus.

**Example 6.8** (The polynomial case). Let  $A = \mathbb{C}[x_1, x_2, \dots, x_n]$ . It is  $n$ -Calabi-Yau since  $\mathrm{Spec}(A) = \mathbb{C}^n$  is  $n$ -Calabi-Yau. Its Koszul dual algebra  $A^! = \mathbf{\Lambda}(\xi_1, \xi_2, \dots, \xi_n)$  has a natural cyclically invariant pairing, which is given by

$$\langle u, v \rangle = \begin{cases} u \wedge v / \xi_1 \cdots \xi_n, & \text{if } |u| + |v| = n, \\ 0, & \text{otherwise.} \end{cases}$$

One immediately sees that,  $\xi_1^* \cdots \xi_n^*$  is a cycle in  $A \otimes A^!$ , representing the volume form both in  $\mathrm{HH}_\bullet(A)$  and  $\mathrm{HH}^\bullet(A^!; A^i)$ . This is completely analogous to the Poisson case (*cf.* Theorem 1.2).

## 7 Calabi-Yau/cyclic algebras and their deformations

In this section we first go over the results of Dolgushev [9] and Willwacher-Calaque [40], saying that for a Calabi-Yau algebra (respectively cyclic algebra), if it is unimodular Poisson (respectively unimodular cyclic Poisson), then its deformation quantization is again Calabi-Yau (respectively cyclic), and then prove Theorems 1.4 and 1.5. The theorem of Willwacher and Calaque was conjectured by Felder and Shoikhet in [13].

### 7.1 Deformation quantization of Calabi-Yau Poisson algebras

Recall that for a Poisson algebra  $A$  with bracket  $\{-, -\}$ , its *deformation quantization*, denoted by  $A_{\hbar}$ , is a  $k[[\hbar]]$ -linear associative product (called the *star-product*) on  $A[[\hbar]]$

$$a * b = a \cdot b + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots,$$

where  $\hbar$  is the formal parameter and  $\mu_i$  are bilinear operators, satisfying

$$\lim_{\hbar \rightarrow 0} \frac{1}{\hbar} (a * b - b * a) = \{a, b\}, \quad \text{for all } a, b \in A.$$

In [19], Kontsevich showed that there is a one-to-one correspondence between the equivalence classes of the star-products and the equivalence classes of Poisson algebra structures on  $A[[\hbar]]$ , where  $A$  is the algebra of smooth functions on the manifold. He also constructed an explicit  $L_\infty$ -quasiisomorphism from the space of polyvector fields to the Hochschild cochain complex; via this map, the Poisson bivector  $\hbar\pi$  on  $A[[\hbar]]$  gives rise to a star-product on  $A[[\hbar]]$ , which is called the *Kontsevich deformation quantization*.

**Theorem 7.1** (Dolgushev [9]). *Let  $A$  be the Calabi-Yau algebra of an affine Calabi-Yau variety. Suppose  $A$  is Poisson, then the deformation quantization of  $A$ , say  $A_{\hbar}$ , is Calabi-Yau over  $k[[\hbar]]$  if and only if  $A$  is unimodular.*

*Proof.* See Dolgushev [9, Theorem 3]. □

**Theorem 7.2** (First part of Theorem 1.5). *Let  $A = k[x_1, \dots, x_n]$  be a unimodular Poisson algebra and  $A_{\hbar}$  be its Kontsevich deformation quantization. Then the following diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A[[\hbar]]) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A[[\hbar]]) \\ \downarrow \cong & & \uparrow \cong \\ \mathrm{HH}^\bullet(A_{\hbar}) & \xrightarrow{\cong} & \mathrm{HH}_{n-\bullet}(A_{\hbar}) \end{array}$$

*is commutative. In particular,*

$$\mathrm{HP}^\bullet(A[[\hbar]]) \cong \mathrm{HH}^\bullet(A_{\hbar})$$

*as Batalin-Vilkovisky algebras.*

Let us remind that in the above theorem (as well as in the rest of this section), the Poisson structure on  $A[[\hbar]]$  is  $\hbar\pi$ . As we mentioned before, this theorem partially answers one of Dolgushev's questions in [9, §7]. Namely, for unimodular Poisson Calabi-Yau algebras, the noncommutative Poincaré duality is preserved under the Kontsevich deformation quantization.

*Proof of Theorem 7.2.* This theorem is an agglomeration of the following results focusing on the Formality Theorem: In [19, Theorem 4.10], Kontsevich showed that on the chain level there is an  $L_\infty$ -quasiisomorphism

$$\mathfrak{X}^\bullet(A)[[\hbar]] \longrightarrow \bar{C}^\bullet(A_\hbar; A_\hbar),$$

and in loc. cit. §8, he sketched that this  $L_\infty$ -morphism also respects the cup product on both sides, which implies that on the cohomology level

$$HP^\bullet(A)[[\hbar]] \longrightarrow HH^\bullet(A_\hbar)$$

is an isomorphism of Gerstenhaber algebras; see also Manchon and Torossian in [28, Théorème 1.1] and by Mochizuki [29, Theorem 1.1] for more details about the signs, etc.

Based on Kontsevich's  $L_\infty$ -quasiisomorphism, one sees that  $\mathfrak{X}^\bullet(A)[[\hbar]]$  acts on  $\Omega^\bullet(A)$  and on  $\bar{C}_\bullet(A; A)$  (action on the latter is through  $\bar{C}^\bullet(A; A)$ ). In [35, Conjecture 5.3.2], Tsygan conjectured that such actions give rise to an  $L_\infty$ -quasiisomorphism of  $L_\infty$ -modules between  $\bar{C}_\bullet(A; A)$  and  $\Omega^\bullet(A)$ . This is known as Tsygan's Formality Conjecture for chains, and is proved by Shoikhet in [31, Theorem 1.3.1]. Shoikhet also conjectured that such  $L_\infty$ -morphism is also compatible with the cup product. Shoikhet's conjecture was proved later by Calaque and Rossi in [4, Theorem A].

Note that there are additional boundary operators on  $\Omega^\bullet(A)$  and  $\bar{C}_\bullet(A; A)$ , which are the de Rham differential and the Connes boundary respectively. One naturally expects the  $L_\infty$ -quasiisomorphism constructed above respects these two operators. This is known as the Cyclic Formality Conjecture for chains, and is proved by Willwacher in [39, Theorem 1.3 and Corollary 1.4].

In summary, the above results combined together show that the two pairs

$$(HP^\bullet(A[[\hbar]]), HP_\bullet(A[[\hbar]])) \quad \text{and} \quad (HH^\bullet(A_\hbar), HH_\bullet(A_\hbar)) \quad (37)$$

are isomorphic as differential calculus. This result has also been proved by Dolgushev, Tamarkin and Tsygan in [10, Corollary 4] and [11, Theorem 9].

Restricting to the Calabi-Yau case, once we know the two versions of Poincaré duality between the pairs (37) are given by capping with the volume form (this volume form is unique up to scalar by the Hochschild-Kostant-Rosenberg theorem), we immediately have that they are further isomorphic as differential calculus with duality. Thus by Theorem 5.3, we have

$$HP^\bullet(A[[\hbar]]) \cong HH^\bullet(A_\hbar)$$

as Batalin-Vilkovisky algebras.

This completes the proof. □

## 7.2 Deformation quantization of cyclic Poisson algebras

**Theorem 7.3** (Felder-Shoikhet [13]; Willwacher-Calaque [40]). *Suppose  $A^! = \Lambda(\xi_1, \dots, \xi_n)$  is a cyclic Poisson algebra. Then the Kontsevich deformation quantization of  $A^!$ , say  $A_\hbar^!$ , is cyclic if and only if  $A^!$  is unimodular.*

The proof of this theorem involves the Cyclic Formality Conjecture for *cochains*. Before proceeding, let us first recall a brief history. The Cyclic Formality Conjecture is due to Kontsevich, and is published in Felder-Shoikhet [13, §1]. It was later proved by Willwacher and Calaque in [40]. It is formulated as follows: Let  $A = C_c^\infty(\mathbb{R}^n)$  be the space of smooth functions on  $\mathbb{R}^n$  with compact support, and fix a constant volume form  $\Omega$  for  $\mathbb{R}^n$ . For a Hochschild cochain of  $A$ , say  $\psi \in \bar{C}^p(A; A)$ , there exists a cochain  $C(\psi) \in \bar{C}^p(A; A)$  such that

$$\int_{\mathbb{R}^n} \psi(f_1, \dots, f_p) \cdot f_{p+1} \cdot \Omega = (-1)^p \int_{\mathbb{R}^n} C(\psi)(f_2, \dots, f_{p+1}) \cdot f_1 \cdot \Omega.$$

As usual, denote by  $\mathcal{D}_{\text{poly}}(\mathbb{R}^n)$  the polydifferential Hochschild cochain space, and let

$$[\mathcal{D}_{\text{poly}}(\mathbb{R}^n)]_{\text{cycl}} := \{\psi \in \mathcal{D}_{\text{poly}}(\mathbb{R}^n) \mid C(\psi) = \psi\}$$

be the cyclically invariant subspace. Then Shoikhet showed that  $[\mathcal{D}_{\text{poly}}(\mathbb{R}^n)]_{\text{cycl}}$  is closed under the Hochschild coboundary and the Gerstenhaber bracket.

On the other hand, denote by  $T_{\text{poly}}^\bullet(\mathbb{R}^n)$  the space of polyvectors on  $\mathbb{R}^n$  (in previous sections it is denoted by  $\mathfrak{X}^\bullet(\mathbb{R}^n)$ ; here we follow the common convention on this subject). One has a divergence operator  $\text{div} : T_{\text{poly}}^\bullet(\mathbb{R}^n) \rightarrow T_{\text{poly}}^{\bullet-1}(\mathbb{R}^n)$  given by the following composition

$$\text{div} : T_{\text{poly}}^p(\mathbb{R}^n) \xrightarrow{-\cap \Omega} \Omega^{n-p}(\mathbb{R}^n) \xrightarrow{d_{\text{dR}}} \Omega^{n-p+1}(\mathbb{R}^n) \xrightarrow{(-\cap \Omega)^{-1}} T_{\text{poly}}^{p-1}(\mathbb{R}^n), \quad (38)$$

which extends to  $T_{\text{poly}}^\bullet(\mathbb{R}^n) \otimes k[u]$  by  $k[u]$ -linearity, where  $u$  is a formal parameter of degree 2.

The Cyclic Formality Conjecture of cochains says that Kontsevich's  $L_\infty$ -quasiisomorphism  $U : T_{\text{poly}}^\bullet(\mathbb{R}^n) \rightarrow \mathcal{D}_{\text{poly}}(\mathbb{R}^n)$  in fact gives a quasiisomorphism

$$U : (T_{\text{poly}}^\bullet(\mathbb{R}^n) \otimes k[u], \text{div}) \longrightarrow ([\mathcal{D}_{\text{poly}}(\mathbb{R}^n)]_{\text{cycl}}, d_{\text{Hoch}}), \quad (39)$$

for any volume form  $\Omega$ . This conjecture was proved by Willwacher and Calaque (see [40, Theorem 2]). As an application, they showed that for any cyclic Poisson structure on  $A$ , its deformation quantization is again cyclic if and only if the Poisson structure is unimodular (see [40, Theorem 37]).

*Proof of Theorem 7.3.* Kontsevich's  $L_\infty$ -quasiisomorphism holds for the graded case, as has been shown in Cattaneo and Felder [5, Appendix] (it has also been used by Shoikhet [32]). The theorem follows verbatim from [40], in particular, Theorem 37 therein.  $\square$

**Remark 7.4.** From cyclic homology theory, the  $L_\infty$ -quasiisomorphism (39) may also be formulated in the form

$$\tilde{U} : (T_{\text{poly}}^\bullet(\mathbb{R}^n) \otimes k[u], \text{div}) \longrightarrow (\mathcal{D}_{\text{poly}}(\mathbb{R}^n) \otimes k[u], d_{\text{Hoch}} + uB^*), \quad (40)$$

where  $B^*$  is the dual Connes operator. This is because the RHS of (39) and of (40) are quasiisomorphic as differential Gerstenhaber algebras (the interested reader may refer to Loday [25, §2.4] for more details).

**Theorem 7.5** (Second part of Theorem 1.5). *Let  $A^! = \mathbf{A}(\xi_1, \dots, \xi_n)$  be a unimodular cyclic Poisson algebra and let  $A_h^!$  be its Kontsevich deformation quantization. Then the following diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A^![[\hbar]]) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^![[\hbar]]; A^![[\hbar]]) \\ \downarrow \cong & & \uparrow \cong \\ \mathrm{HH}^\bullet(A_h^!) & \xrightarrow{\cong} & \mathrm{HH}^{\bullet-n}(A_h^!; A_h^!) \end{array}$$

is commutative. In particular,

$$\mathrm{HH}^\bullet(A_h^!) \cong \mathrm{HP}^\bullet(A^![[\hbar]])$$

as Batalin-Vilkovisky algebras.

*Proof.* Theorem 7.3 together with Remark 7.4 implies that

$$(\mathrm{HP}^\bullet(A^![[\hbar]]), \mathrm{HP}^{\bullet-n}(A^![[\hbar]]; A^![[\hbar]])) \quad \text{and} \quad (\mathrm{HH}^\bullet(A_h^!), \mathrm{HH}^{\bullet-n}(A_h^!; A_h^!))$$

are isomorphic as differential calculus with duality, and hence by Theorem 5.3 the conclusion follows.  $\square$

### 7.3 Identification of two Poincaré dualities

We are now reaching the following main result of the paper:

**Theorem 7.6** (Theorem 1.4). *Suppose  $A = k[x_1, \dots, x_n]$  is a unimodular quadratic Poisson algebra. Denote by  $A^!$  the Koszul dual algebra of  $A$ , and by  $A_h$  and  $A_h^!$  the Kontsevich deformation quantization of  $A$  and  $A^!$  respectively. Then the following diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A[[\hbar]]) & \xrightarrow{\cong} & \mathrm{HP}^\bullet(A^![[\hbar]]) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^\bullet(A_h) & \xrightarrow{\cong} & \mathrm{HH}^\bullet(A_h^!) \end{array} \tag{41}$$

is a commutative diagram of Batalin-Vilkovisky algebra isomorphisms.

*Proof.* By Shoikhet's [32, Theorem 0.3],  $A_h$  and  $A_h^!$  are Koszul dual algebras over  $k[[\hbar]]$ , hence, Theorem 7.6 follows from a combination of Theorems 5.5, 7.2, 7.5, and 6.6.  $\square$

### 7.4 Twisted Poincaré duality for Poisson algebras

For a general associative algebra, say  $A$ , it may not be Calabi-Yau, and therefore there may not exist Poincaré duality between  $\mathrm{HH}^\bullet(A)$  and  $\mathrm{HH}_\bullet(A)$ . Inspired by Van den Bergh [37], Brown and Zhang [2] introduced the so-called “twisted Poincaré duality” for associative algebras. That is, for such  $A$ , keeping its left  $A$ -module structure (the multiplication) as usual, the right  $A$ -module structure of  $A$  is the multiplication composed with an automorphism  $\sigma : A \rightarrow A$ . Denote such  $A$ -bimodule by  $A_\sigma$ , then Brown and Zhang showed that for a lot of algebras, there exists a twisted Poincaré duality  $\mathrm{HH}^\bullet(A) \cong \mathrm{HH}_{n-\bullet}(A; A_\sigma)$  for some  $n \in \mathbb{N}$  (cf. [2, Corollary 5.2]). In this case  $A$  is called a *twisted Calabi-Yau* of dimension  $n$ .

Such phenomenon also occurs for Poisson algebras. Namely, not all Poisson algebras are unimodular, and hence there may not exist isomorphism between  $HP^\bullet(A)$  and  $HP_\bullet(A)$ . In [22, 27, 42, 43], the authors studied the so-called twisted Poincaré duality for Poisson algebras, similar to that of associative algebras. They also studied some comparisons with twisted Calabi-Yau algebras. However, it would be very interesting to study the relationships between the deformation quantization of twisted unimodular Poisson algebras and twisted Calabi-Yau algebras, and obtain a theorem similar to Theorem 7.6 in this twisted case.

## References

- [1] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* 9 (1996) 473–527.
- [2] K. Brown, J. J. Zhang, Dualising complexes and twisted Hochschild (co)homology for noetherian Hopf algebras, *J. Algebra* 320 (2008) 1814–1850.
- [3] J. Brylinski, A differential complex for Poisson manifolds, *J. Differential Geom.* 28 (1988) 93–114.
- [4] D. Calaque, C. A. Rossi, Compatibility with cap-products in Tsygan’s formality and homological Duflo isomorphism, *Lett. Math. Phys.* 95 (2011) 135–209.
- [5] A. S. Cattaneo, G. Felder, Relative formality theorem and quantisation of coisotropic submanifolds, *Adv. Math.* 208 (2007) 521–548.
- [6] X. Chen, S. Yang, G. Zhou, Batalin-Vilkovisky algebras and the non-commutative Poincaré duality of Koszul Calabi-Yau algebras, *J. Pure Appl. Algebra* 220 (2016) 2500–2532.
- [7] Yu. Daletskii, I. Gelfand, B. Tsygan, On a variant of noncommutative differential geometry, *Dokl. Akad. Nauk SSSR* 308 (1989) 1239–1297 (in Russian), translation in *Soviet Math. Dokl.* 40 (1990) 422–426.
- [8] L. de Thanhoffer de Völcsey, M. Van den Bergh, Calabi-Yau Deformations and Negative Cyclic Homology, arXiv:1201.1520v4, to appear in *Journal of Noncommutative Geometry*.
- [9] V. A. Dolgushev, The Van den Bergh duality and the modular symmetry of a Poisson variety, *Selecta Math. (N.S.)* 14 (2009) 199–228.
- [10] V. A. Dolgushev, D. E. Tamarkin, B. L. Tsygan, Formality of the homotopy calculus algebra of Hochschild (co)chains, arXiv:0807.5117.
- [11] V. A. Dolgushev, D. E. Tamarkin, B. L. Tsygan, Formality theorems for Hochschild complexes and their applications, *Lett. Math. Phys.* 90 (2009) 103–136.
- [12] P. Etingof, V. Ginzburg, Noncommutative del Pezzo surfaces and Calabi-Yau algebras, *J. Eur. Math. Soc.* 12 (2010) 1371–1416.
- [13] G. Felder, B. Shoikhet, Deformation quantization with traces, *Lett. Math. Phys.* 53 (2000) 75–86.
- [14] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. of Math.* 78 (1963) 267–288.



- [15] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theories, *Comm. Math. Phys.* 159 (1994) 265–285.
- [16] V. Ginzburg, Calabi-Yau algebras, [arXiv:0612139v3](https://arxiv.org/abs/0612139v3).
- [17] E. Herscovich, Hochschild (co)homology and Koszul duality, [arXiv:1405.2247](https://arxiv.org/abs/1405.2247).
- [18] J. Huebschmann, Poisson cohomology and quantization, *J. reine angew. Math.* 408 (1990) 57–113.
- [19] M. Kontsevich, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* 66 (2003) 157–216.
- [20] J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, *Astérisque*, numéro hors série, 257–271 (1985).
- [21] T. Lambre, Dualité de Van den Bergh et Structure de Batalin-Vilkovisky sur les algèbres de Calabi-Yau, *J. Noncom. Geom.* 3 (2010) 441–457.
- [22] S. Launois, L. Richard, Twisted Poincaré duality for some quadratic Poisson algebras, *Lett. Math. Phys.* 79 (2007) 161–174.
- [23] C. Laurent-Gengoux, A. Pichereau and P. Vanhaecke, *Poisson structures*, Grundle Math. Wiss. 347. Springer, Heidelberg, 2013.
- [24] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geom.* 12 (1977) 253–300.
- [25] J.-L. Loday, *Cyclic homology*, 2nd edition, Grundle Math. Wiss. 301, Springer-Verlag, Berlin, 1998.
- [26] J.-L. Loday, B. Vallette, *Algebraic Operads*, Grundle Math. Wiss. 346, Springer-Verlag (2012).
- [27] J. Luo, S.-Q. Wang, Q.-S. Wu, Twisted Poincaré duality between Poisson homology and Poisson cohomology, *J. Algebra* 442 (2015) 484–505.
- [28] D. Manchon, C. Torossian, Cohomologie tangente et cup-produit pour la quantification de Kontsevich, *Ann. Math. Blaise Pascal* 10 (2003) 75–106. Erratum: *Ann. Math. Blaise Pascal* 11 (2004) 129–130.
- [29] T. Mochizuki, On the morphism of Duflo-Kirillov type, *J. Geom. Phys.* 41 (2002) 73–113.
- [30] B. Pym, Quantum deformations of projective three-space, *Adv. Math.* 281 (2015) 1216–1241.
- [31] B. Shoikhet, A proof of the Tsygan formality conjecture for chains, *Adv. Math.* 179 (2003) 7–37.
- [32] B. Shoikhet, Koszul duality in deformation quantization and Tamarkin’s approach to Kontsevich formality, *Adv. Math.* 224 (2010) 731–771.
- [33] D. Tamarkin, B. Tsygan, The ring of differential operators on forms in noncommutative calculus, *Graphs and patterns in mathematics and theoretical physics*, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 105–131.
- [34] T. Tradler, The Batalin-Vilkovisky algebra on Hochschild cohomology induced by infinity inner products, *Ann. Inst. Fourier (Grenoble)* 58 (2008) 2351–2379.

- [35] B. Tsygan, Formality conjectures for chains, Differential topology, infinite-dimensional Lie algebras, and applications, 261–274, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999.
- [36] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra 195 (1997) 662–679.
- [37] M. Van den Bergh, A relation between Hochschild homology and cohomology for Gorenstein rings, Proc. Amer. Math. Soc. 126 (1998) 1345–1348.
- [38] A. Weinstein, The modular automorphism group of a Poisson manifold, J. Geom. Phys. 23 (1997) 379–394.
- [39] T. Willwacher, Formality of cyclic chains, Int. Math. Res. Not. 17 (2011) 3939–3956.
- [40] T. Willwacher, D. Calaque, Formality of cyclic cochains, Adv. Math. 231 (2012) 624–650.
- [41] P. Xu, Gerstenhaber algebras and BV-algebras in Poisson geometry, Comm. Math. Phys. 200 (1999) 545–560.
- [42] C. Zhu, F. Van Oystaeyen, Y. Zhang, On (co)homology of Frobenius Poisson algebras, J. K-Theory 14 (2014) 371–386.
- [43] C. Zhu, Twisted Poincaré duality for Poisson homology and cohomology of affine Poisson algebras, Proc. Amer. Math. Soc. 143 (2015) 1957–1967.